Lecture notes for 12.006J/18.353J/2.050J, Nonlinear Dynamics: Chaos

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October 31, 2022

Contents

1 Introduction to strange attractors

References: Bergé et al. [1], Strogatz [2], Abraham and Shaw [3]

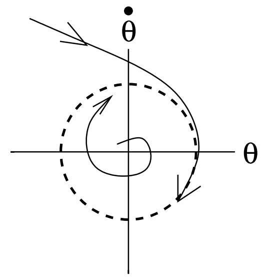
Thus far, we have studied classical attractors: fixed points and limit cycles. In this lecture we begin our study of *strange attractors*. We emphasize their generic features.

1.1 Dissipation and attraction

Our studies of oscillators have revealed explicitly how forced systems can reach a stationary (yet dynamic) state characterized by an energy balance:

$$\text{average energy supplied} = \text{average energy dissipated}$$

An example is a limit cycle:



Initital conditions inside or outside the limit cycle always evolve to the limit cycle.

Limit cycles are a specific way in which

$$\text{dissipation} \Rightarrow \text{attraction.}$$

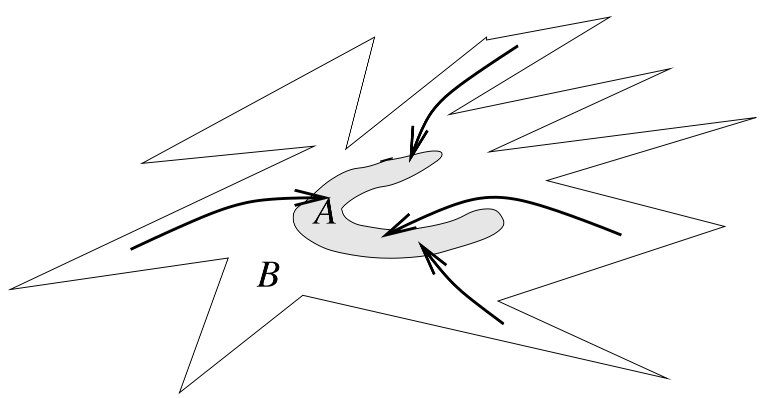
More generally, we have an n-dimensional flow

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \vec{F}[\vec{x}(t)], \qquad \vec{x} \in \mathbb{R}^n \tag{1}$$

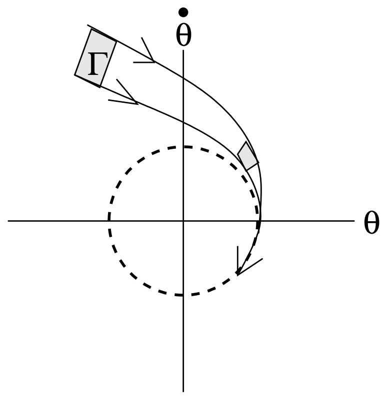
Assume that the flow $\vec{x}(t)$ is dissipative, with attractor A.

Properties of the attractor A:

- A is invariant with flow (i.e., it does not change with time).
- A is contained within B, the *basin of attraction.* B is that part of phase space from which all initial conditions lead to A as $t \to \infty$:



- A has dimension $d < n$.
  Consider, for example, the case of a limit cycle:



The surface Γ is reduced by the flow to a line segment on the limit cycle (the attractor). Here

$$\begin{aligned} d &= \text{attractor dimension} = 1 \\ n &= \text{phase-space dimension} = 2. \end{aligned}$$

This phenomenon is called *reduction of dimensionality*.

Consequence: loss of information on initial conditions.

We have already quantified volume contraction. Given an initial volume V evolving according to the flow (1), the Lie derivative tells us that V changes as

$$\frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}t} = \nabla \cdot \dot{\vec{x}} = \sum_i^n \frac{\partial \dot{x}_i}{\partial x_i}$$

As we showed earlier, dissipation yields *volume contraction*; i.e.,

$$\frac{\mathrm{d}V}{\mathrm{d}t} < 0.$$

Consequently, the attractor cannot have n-dimensional volumes, so $d < n$.

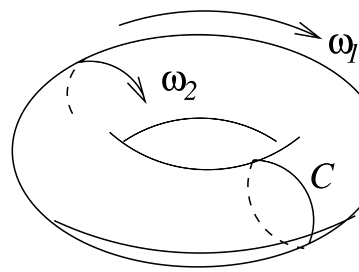
What, then, is the dimension of the attractor?

We proceed by example, by considering the case $d = 2$.

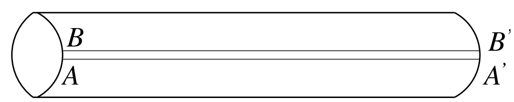
1.2 Attractors with $d = 2$

What happens when d (the dimension of the attractor) is 2?

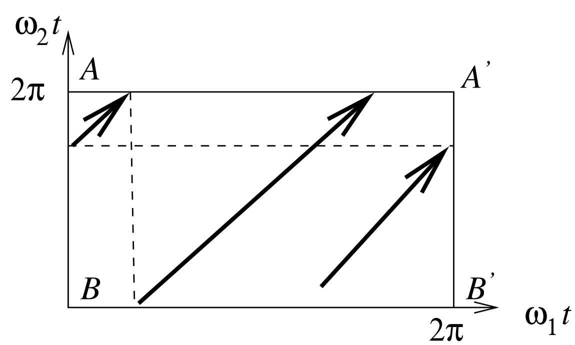
Assume a quasiperiodic attractor on a torus T^2:



Cut the torus on a small circle C and open it:



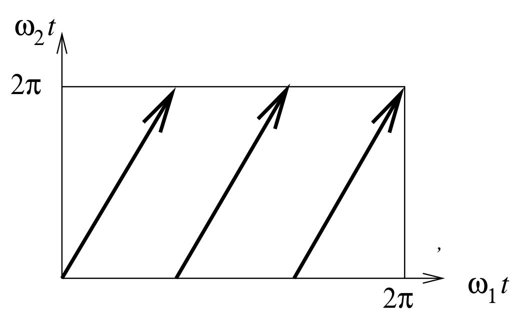
Finally, cut the long way, from A to A', and open it again:



Note the parallel trajectories.

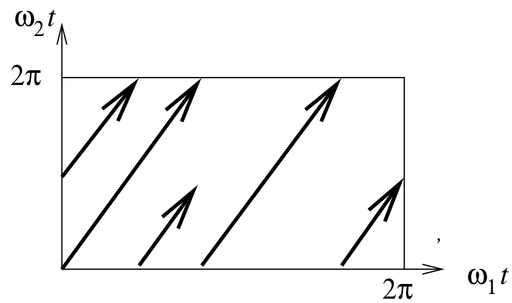
As usual, the quasiperiodic flows are characterized by two cases: ω_1/ω_2 rational or irrational.

- Rational. Consider, e.g., $\omega_1/\omega_2 = 1/3$:



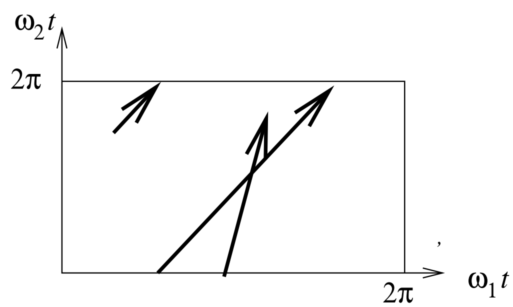
The trajectory repeats itself exactly every three times around the 2-axis, or each time around the 1-axis.

- Irrational.



The trajectories densely fill the plane.

Determinism forbids non-parallel trajectories, because they would cross:

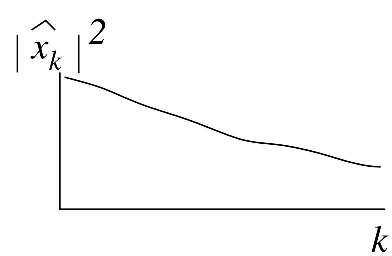


Thus a torus T^2 can only be a periodic or quasiperiodic attractor.

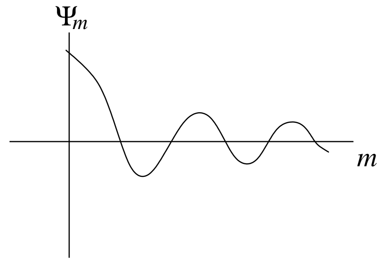
The attractor cannot be aperiodic if $d = 2$.

1.3 Aperiodic attractors

We have already shown that the power spectrum of an aperiodic signal $x(t)$ is continuous:



And the autocorrelation $\Psi_m = \langle x_j x_{j+m} \rangle$ has finite width:

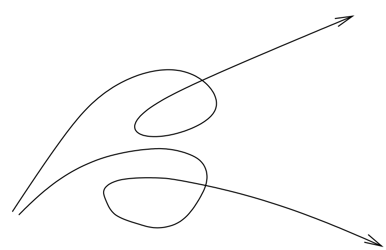


The finite width of Ψ_m implies that knowledge of no finite interval of $x(t)$ allows prediction of all future $x(t)$.

This "unpredictability" is associated with what we call "chaos." We seek, however, a more precise definition of chaos.

On an aperiodic attractor, small differences in initial conditions *on the attractor* lead at later times to large differences, *still on the attractor*.

In phase space, trajectories on an aperiodic attractor can diverge, e.g.,



We shall see that the divergence of trajectories is exponential in time.

This phenomenon is called *sensitivity to initial conditions* (SIC). It definitively identifies *chaos*, i.e., a chaotic attractor.

Note that, despite the precision of this definition, we are left with an apparent conundrum: simultaneously we have

- attraction, such that trajectories *converge*.
- sensitivity to initial conditions, such that trajectories *diverge*.

The conundrum is solved by noting that trajectories converge *to* the attractor, but diverge *on* the attractor.

Note further that divergence on the attractor implies that the attractor dimension

$$d > 2,$$

since phase tractories cannot diverge in two dimensions.

Thus we conclude that an aperiodic (chaotic) attractor must have phase space dimension

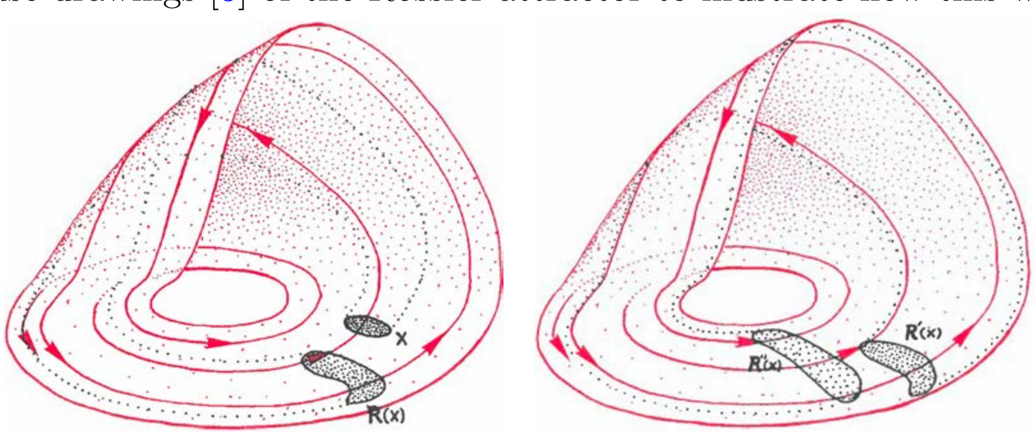
$$n \geq 3.$$

Assume $n = 3$. How may trajectories converge, but still remain bounded on an attractor?

The trajectories are successively *stretched* (by SIC) and *folded* (thus remaining bounded).

1.4 Example: Rössler attractor

We use drawings [3] of the Rössler attractor to illustrate how this works:

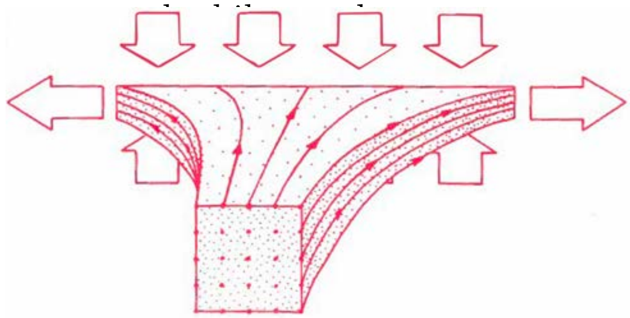


Abraham & Shaw [3]

See image credit on Page 12.

- Trajectories diverge in plane by spiralling out (stretching).
- Trajectories leave plane.
- Trajectories return to plane (folding), back to center of spiral.

At the same time, we must have volume contraction. One dimension can



Abraham & Shaw [3] See image credit on Page 12.

Let's consider the stretching and folding in more detail. The Rössler attractor reads

$$\begin{aligned}\dot{x} &= -y - z\\ \dot{y} &= x + ay\\ \dot{z} &= b + z(x - c)\end{aligned}$$

where we assume

$$a > 0.$$

Assume z and $\dot{z}$ are small. Then in the x, y plane the system is approximated by

$$\begin{aligned}\dot{x} &= -y\\ \dot{y} &= x + ay.\end{aligned}$$

Then

$$\ddot{x} = -\dot{y} = -x + a\dot{x}$$

yielding the *negatively damped* oscillator

$$\ddot{x} - a\dot{x} + x = 0.$$

Consequently the trajectories spiral out of the origin.

How is the spreading confined? From the equation for $\dot{z}$, we see that, for small b,

$$\begin{aligned}x < c &\Rightarrow \dot{z} < 0\\ x > c &\Rightarrow \dot{z} > 0\end{aligned}$$

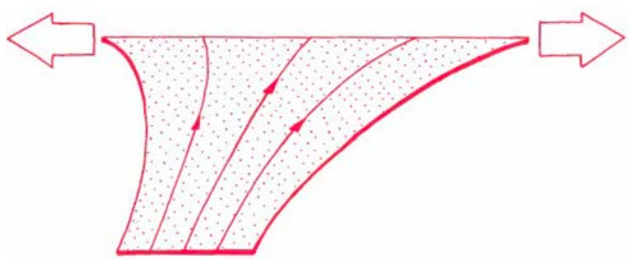
Thus we expect trajectories to behave as follows:

- Divergence from the origin creates $x > c$.
- $x > c \Rightarrow z$ increases $\Rightarrow x$ decreases.
- Eventually x decreases such that $x < c$.
- Then $x < c \Rightarrow z$ decreases $\Rightarrow$ back in the x, y plane.
- The process repeats.

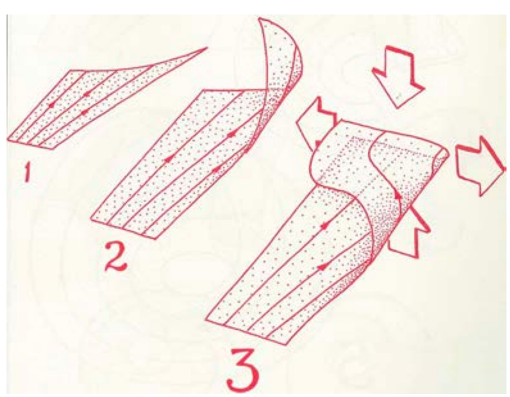
Thus we have

- *stretching*, from the outward spiral; and
- *folding*, from the feedback of z into x.

A sequence of figures shows how endless divergence occurs in bounded space.

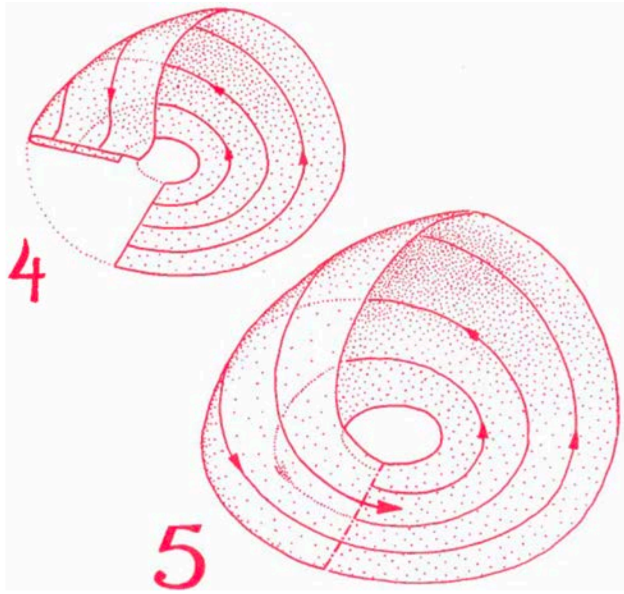


Abraham & Shaw [3] See image credit on Page 12.



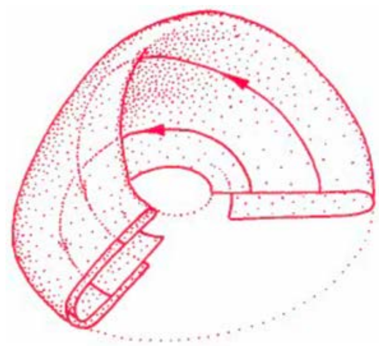
Abraham & Shaw [3] See image credit on Page 12.

Trajectories never close exactly as a surface, but more like filo dough:

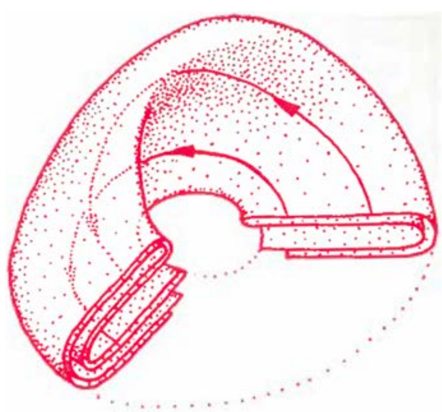


Abraham & Shaw [3] See image credit on Page 12.

Another iteration of stretching and folding looks like this:

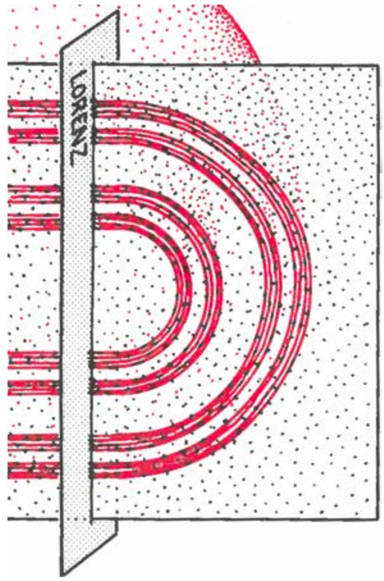


Abraham & Shaw [3] See image credit on Page 12.



Abraham & Shaw [3] See image credit on Page 12.

Each time we stretch and fold, we create layers of layers. And when we slice it with a Poincar/'e section, we would see gaps within gaps, and thus a



Abraham & Shaw [3] See image credit on Page 12.

We'll say more about this later, after we describe its earliest observation in the Lorenz attractor.

1.5 Conclusion

We arrive at the following conclusions:

- Aperiodic attractors must have

$$d > 2.$$

- Since dissipation contracts volumes,

$$d < n,$$

  where n is the dimension of the phase space.

- Suppose $n = 3$. Then a chaotic attractor must have

$$2 < d < 3.$$

How can $2 < d < 3$? The attractor has a fractional, or *fractal* dimension. We shall look more closely at this later.

For now, we conclude that chaotic attractors have three properties:

- Attraction
- Sensitivity to initial conditions
- Non-integer fractal dimension

The combination of these three properties defines a *strange attractor*. The "strangeness" arises not so much from each individual property but their combined presence.

Next we study the most celebrated strange attractor—the *Lorenz attractor*.

References

1. Bergé, P., Pomeau, Y. & Vidal, C. *Order within Chaos: Towards a Deterministic Approach to Turbulence* (John Wiley and Sons, New York, 1984).
2. Strogatz, S. *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering* (CRC Press, 2018).
3. Abraham, R. H. & Shaw, C. D. *Dynamics-The Geometry of Behavior: Part 2: Chaotic Behavior* (Aerial Press, Incorporated, 1984).

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