Lecture notes for 12.006J/18.353J/2.050J, Nonlinear Dynamics: Chaos

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Contents

1 Forced oscillators and limit cycles

1.1 General remarks

How may we describe a forced oscillator?
The linear equation

$$\ddot{\theta} + \gamma\dot{\theta} + \omega^2\theta = 0 \tag{1}$$

is in general inadequate. Why?

Linearity $\Rightarrow$ if $\theta(t)$ is a solution, then so is $\alpha\theta(t)$, α real. This is incompatible with bounded oscillations (i.e., $\theta_{\max} < \pi$).

We therefore introduce an equation with

- a nonlinearity; and
- an energy source that compensates viscous damping.

1.2 Van der Pol equation

Consider a damping coefficient $\gamma(\theta)$ such that

$$\gamma(\theta) > 0 \qquad \text{for } |\theta| \text{ large}$$

$$\gamma(\theta) < 0 \qquad \text{for } |\theta| \text{ small}$$

Express this in terms of θ^2:

$$\gamma(\theta) = \gamma_0 \left(\frac{\theta^2}{\theta_0^2} - 1 \right)$$

where $\gamma_0 > 0$ and θ_0 is some reference amplitude.

Now, obviously,

$$\gamma > 0 \qquad \text{for } \theta^2 > \theta_0^2$$
$$\gamma < 0 \qquad \text{for } \theta^2 < \theta_0^2$$

Substituting γ into (1), we get

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \gamma_0 \left(\frac{\theta^2}{\theta_0^2} - 1 \right) \frac{\mathrm{d}\theta}{\mathrm{d}t} + \omega^2\theta = 0$$

This equation is known as the *van der Pol equation*. It was introduced in the 1920's as a model of nonlinear electric circuits used in the first radios.

In van der Pol's (vaccum tube) circuits,

- high current $\Longrightarrow$ positive (ordinary) resistance; and
- low current $\Longrightarrow$ negative resistance.

The basic behavior: large oscillations decay and small oscillations grow.

We shall examine this system in some detail. First, we write it in non-dimensional form.

We define new units of time and amplitude:

- unit of time $= 1/\omega$
- unit of amplitude $= \theta_0$.

We transform

$$\begin{aligned} t &\rightarrow t'/\omega \\ \theta &\rightarrow \theta'\theta_0 \end{aligned}$$

where θ' and t' are non-dimensional.

Substituting above, we obtain

$$\omega^2 \frac{\mathrm{d}^2\theta'}{\mathrm{d}t'^2}\theta_0 + \gamma_0 \left[\left(\frac{\theta'\theta_0}{\theta_0} \right)^2 - 1 \right] \frac{\mathrm{d}\theta'}{\mathrm{d}t'}\omega\theta_0 + \omega^2\theta'\theta_0 = 0$$

Divide by $\omega^2\theta_0$:

$$\frac{\mathrm{d}^2\theta'}{\mathrm{d}t'^2} + \frac{\gamma_0}{\omega}\left(\theta'^2 - 1\right)\frac{\mathrm{d}\theta'}{\mathrm{d}t'} + \theta' = 0$$

Now define the dimensionless control parameter

$$\mu = \frac{\gamma_0}{\omega} > 0.$$

Finally, drop primes to obtain

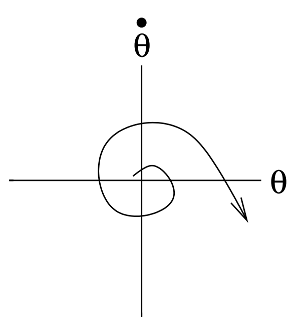
$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \mu(\theta^2 - 1)\frac{\mathrm{d}\theta}{\mathrm{d}t} + \theta = 0. \tag{2}$$

What can we say about the phase portraits?

- When the amplitude of oscillations is small ($\theta_{\max} < 1$), we have

  $$\mu(\theta_{\max}^2 - 1) < 0 \Rightarrow \text{negative damping}$$

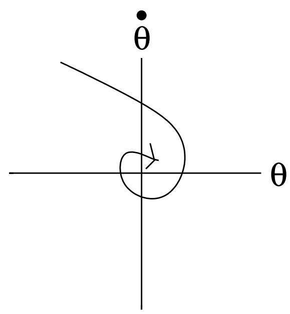
  Thus trajectories spiral outward:



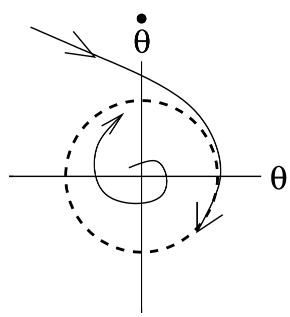
- But when the amplitude of oscillations is large ($\theta_{\text{max}} > 1$),

$$\mu(\theta_{\text{max}}^2 - 1) > 0 \Rightarrow \text{positive damping}$$

The trajectories spiral inward:



Intuitively, we expect a closed trajectory between these two extreme cases:



This closed trajectory is called a *limit cycle*.

For $\mu > 0$, the limit cycle is an *attractor* (and is stable).

This is a new kind of attractor. Instead of representing a single fixed point, it represents stable oscillations.

Examples of such stable oscilations abound in nature: heartbeats, circadian (daily) cycles in body temperature, etc. Small perturbations always return to the standard cycle.

What can we say about the limit cycle of the van der Pol equation?

Chapter 7 of Strogatz [1] shows how one can prove the existence and stability of limit cycles.

In the present case, we can make substantial progress with a simple energy balance argument.

1.3 Energy balance for small μ

References: Bergé et al. [2]

Let $\mu \to 0$, and take θ small. Using our previous expression for energy in the pendulum, the non-dimensional energy is

$$E(\theta, \dot{\theta}) = \frac{1}{2}(\dot{\theta}^2 + \theta^2)$$

The time variation of energy is

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{1}{2}(2\dot{\theta}\ddot{\theta} + 2\dot{\theta}\theta)$$

From the van der Pol equation (2), we have

$$\ddot{\theta} = -\mu(\theta^2 - 1)\dot{\theta} - \theta.$$

Substituting this into the expression for $\mathrm{d}E/\mathrm{d}t$, we obtain

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}t} &= \mu\dot{\theta}^2(1 - \theta^2) - \theta\dot{\theta} + \theta\dot{\theta} && (3) \\ &= \mu\dot{\theta}^2(1 - \theta^2) && (4) \end{aligned}$$

Now define the average of a function $f(t)$ over one period of the oscillation:

$$\overline{f} \equiv \frac{1}{2\pi}\int_{t_0}^{t_O + 2\pi} f(t)\mathrm{d}t.$$

Then the average energy variation over one period is

$$\overline{\frac{\mathrm{d}E}{\mathrm{d}t}} = \frac{1}{2\pi}\int_{t_0}^{t_0 + 2\pi} \frac{\mathrm{d}E}{\mathrm{d}t}\mathrm{d}t.$$

Substituting equation (4) for $\mathrm{d}E/\mathrm{d}t$, we obtain

$$\overline{\frac{\mathrm{d}E}{\mathrm{d}t}} = \mu\overline{\dot{\theta}^2} - \mu\overline{\dot{\theta}^2\theta^2}.$$

In steady state, the production of energy, $\mu\overline{\dot\theta^2}$, is exactly compensated by the dissipation of energy, $\mu\overline{\dot\theta^2\theta^2}$. Thus

$$\mu\overline{\dot\theta^2} = \mu\overline{\dot\theta^2\theta^2}$$

or

$$\overline{\dot\theta^2} = \overline{\dot\theta^2\theta^2}.$$

Now consider the limit $\mu \to 0$ (from above).
We know the approximate solution:

$$\theta(t) = \rho \sin t,$$

i.e., simple sinusoidal oscillation of unknown amplitude ρ.

We proceed to calculate ρ from the energy balance.

The average rate of energy production is

$$\overline{\dot\theta^2} \simeq \frac{1}{2\pi}\int_{t_0}^{t_0+2\pi} \rho^2 \cos^2 t \mathrm{d}t = \frac{1}{2}\rho^2.$$

The average rate of energy dissipation is

$$\overline{\dot\theta^2\theta^2} \simeq \frac{1}{2\pi}\int_{t_0}^{t_0+2\pi} \rho^4 \sin^2 t \cos^2 t \mathrm{d}t = \frac{1}{8}\rho^4.$$

The energy balance argument gives

$$\frac{1}{2}\rho^2 = \frac{1}{8}\rho^4.$$

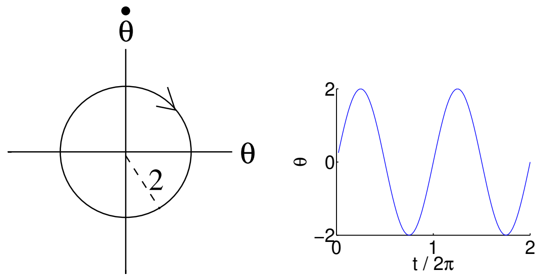
Therefore

$$\rho = 2.$$

We thus find that, independent of $\mu = \gamma_0/\omega$, we have the following approximate solution for $\mu \ll 1$:

$$\theta(t) \simeq 2\sin t.$$

That is, we have a limit cycle with an amplitude of 2 dimensionless units. Graphically,



1.4 Limit cycle for μ large

Reference: Strogatz [1], Ch. 7

The case of μ large requires a different analysis.

First, we introduce an unconventional set of phase plane variables (not $\dot{x} = y, \dot{y} = \ldots$). That is, the phase plane coordinates will not be θ and $\dot{\theta}$.

Recall the van der Pol equation (2), but write in terms of $x = \theta$:

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0. \tag{5}$$

Notice that

$$\ddot{x} + \mu\dot{x}(x^2 - 1) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\dot{x} + \mu\left(\frac{1}{3}x^3 - x\right)\right].$$

Let

$$F(x) = \frac{1}{3}x^3 - x \tag{6}$$

and

$$w = \dot{x} + \mu F(x). \tag{7}$$

Then, using (6) and (7), we have

$$\dot{w} = \ddot{x} + \mu\dot{x}(x^2 - 1).$$

Substituting the van der Pol equation (5), this gives

$$\dot{w} = -x \tag{8}$$

Now rearrange equation (7) to obtain

$$\dot{x} = w - \mu F(x) \tag{9}$$

We have thus parameterized the system by x and w. However we make one more change of variable. Write

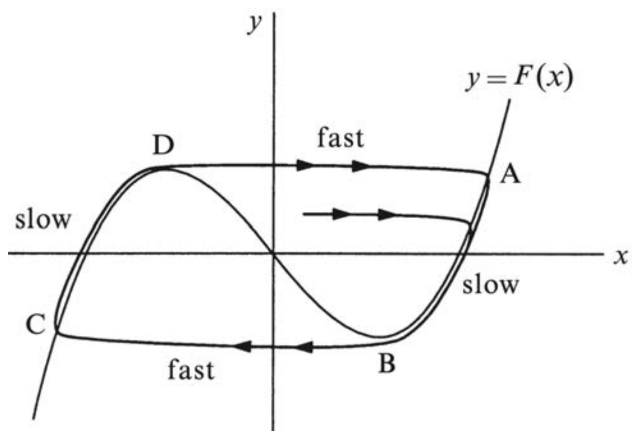
$$y = w/\mu.$$

Then (8) and (9) become

$$\begin{aligned} \dot{x} &= \mu[y - F(x)] \qquad (10) \\ \dot{y} &= -\frac{1}{\mu}x \qquad (11) \end{aligned}$$

Now consider a trajectory in the x-y plane.

First, draw the *nullcline* for x, that is, the curve showing where $\dot{x} = 0$. This is the cubic curve $y = F(x)$.



Strogatz [1], Fig. 7.5.1

Note that the $\dot{y} = 0$ nullcline is the $x = 0$, i.e., the y-axis.

Now imagine a trajectory starting not too close to $y = F(x)$, i.e.. suppose

$$y - F(x) \sim 1.$$

Then from the equations of motion (10) and (11),

$$\begin{aligned} \dot{x} &\sim \mu \gg 1 \\ \dot{y} &\sim 1/\mu \ll 1 \qquad \text{assuming } x \sim 1. \end{aligned}$$

Thus the horizontal velocity is large, the vertical velocity is small, and trajectories move horizontally. Indeed the vertical velocity vanishes on the y-nullcline ($x = 0$).

Eventually the trajectory is so close to $y = F(x)$ such that

$$y - F(x) \sim \frac{1}{\mu^2}$$

implying that

$$\dot{x} \sim \dot{y} \sim \frac{1}{\mu}.$$

Thus the trajectory crosses the nullcline (vertically, since $\dot{x} = 0$ on the nullcline).

Then $\dot{x}$ changes sign, we still have $\dot{x} \sim \dot{y} \sim 1/\mu$, and the trajectories crawl slowly along the nullcline.

What happens at the knee (the minimum of $F(x)$)?
The trajectories jump sideways again, as may be inferred from the symmetry $x \to -x, \;\; y \to -y$.

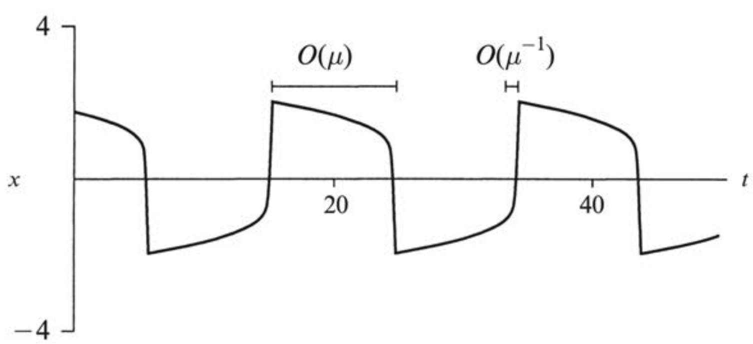
The trajectory closes to form the limit cycle.

Summary: The dynamics has two widely separated time scales:

- The crawls: $\Delta t \sim \mu \qquad (\dot{x} \sim 1/\mu)$
- The jumps: $\Delta t \sim 1/\mu \qquad (\dot{x} \sim \mu)$

Such systems are called slow-fast systems.

A time series of $x(t) = \theta(t)$ shows a classic *relaxation oscillation*:



Strogatz [1], Fig. 7.5.2

Relaxation oscillations are periodic processes with two time scales: a slow buildup is followed by a fast discharge.

Examples include

- stick-slip friction (earthquakes, avalanches, bowed violin strings, etc.)
- nerve cells, heart beats (large literature in mathematical biology...)

1.5 A final note

Limit cycles exist only in nonlinear systems. Why?

A linear system $\dot{\vec{x}} = A\vec{x}$ can have closed periodic orbits, but not an *isolated* orbit.

That is, linearity requires that if $\vec{x}(t)$ is a solution, so is $\alpha\vec{x}(t)$, $\alpha \neq 0$.

Thus the amplitude of a periodic cycle in a linear system depends on the initial conditions.

The amplitude of a limit cycle, however, is independent of the initial conditions.

References

1. Strogatz, S. *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering* (CRC Press, 2018).
2. Bergé, P., Pomeau, Y. & Vidal, C. *Order within Chaos: Towards a Deterministic Approach to Turbulence* (John Wiley and Sons, New York, 1984).