14.271 Ellison/Salz
Fall 2022

Problem Set 4 Solutions
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October 19, 2022

Question 1

(a) In lecture, Glenn proved that Stahl's model has no pure strategy Nash equilibrium. This proof relied on an "undercutting" argument: if firms set prices $p_i = p > c$ for all i, then any firm could increase profits by deviating to $p - \epsilon$. Such undercutting behavior may not be possible in this setting, as firms are restricted to a discrete set of possible prices. If, for instance, $p_1 = p_2 = \frac{1}{4}$, neither firm can deviate to $\frac{1}{4} - \epsilon$ because $\frac{1}{4} - \epsilon$ is not in their choice set.

(b) We seek to characterize the set of parameters (s, μ) such that $p_1 = p_2 = \frac{1}{4}$ is a mutual best response.

If $p_1 = p_2 = \frac{1}{4}$, the consumer buys $\frac{3}{4}$ units of the good from firm i with probability $\frac{1}{2}$. Profits are therefore $\pi_1 = \pi_2 = \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{32}$. If firm 1 deviates to $p_1 = 0$, she earns profit 0. If she deviates to some $p_1 > \frac{1}{4}$, then she sells to none of the μ shoppers, nor to any of the $\frac{1}{2}(1-\mu)$ costly searchers who visit firm 2 first; firm 1 may, however, sell to the $\frac{1}{2}(1-\mu)$ costly searchers who visit firm 1 first. These consumers will purchase from firm 1 iff

$$\begin{aligned}\frac{1}{2}(1-p_1)^2 &\geq \frac{1}{2}\left(1 - \frac{1}{4}\right)^2 - s \\ &= \frac{7}{32} - s \qquad (1)\end{aligned}$$

Here, we use the fact that a consumer's net utility from purchasing at price p is $u(p) \equiv \int_0^{1-p}(1-x)dx - p(1-p) = \frac{1}{2}(1-p)^2$.

Setting $p_1 = \frac{1}{2}$, (1) becomes

$$\begin{aligned}\frac{3}{8} &\geq \frac{7}{32} - s \\ s &\geq \frac{5}{32}\end{aligned}$$

Suppose $s \geq \frac{5}{32}$ and firm 1 set $p_1 = \frac{1}{2}$. Then, firm 1's profit is $\frac{1}{8}(1-\mu)$. A deviation from $p_1 = \frac{1}{4}$ to $p_1 = \frac{1}{2}$ is profitable iff $\mu \leq \frac{1}{4}$.

Note that we need not consider deviations to $\frac{3}{4}$ or 1. Intuitively, the fact that these prices are above the price set by a monopolist ($\frac{1}{2}$) indicate that they cannot be optimal in this duopoly setting; $\frac{3}{4}$ and 1 are strictly dominated by $\frac{1}{2}$.

So, $p_1 = \frac{1}{4}$ is a best response to $p_2 = \frac{1}{4}$ if either (i) $s < \frac{5}{32}$ or (ii) $\mu > \frac{1}{4}$. By symmetry, these conditions guarantee that $p_1 = p_2 = \frac{1}{4}$ is a mutual best response, and therefore a pure strategy Nash equilibrium.

[1]Solutions based on those of Adam Harris, Anton Popov, and Sam Grondahl.

(c) Setting a price of either 0 or 1 results in the firm earning zero profits. Recall that in a mixed-strategy Nash equilibrium, if a firm is mixing over a set of prices P (i.e. $p \in P \Rightarrow \sigma^*(p) > 0$), then the firm's expected profit must be equalized across all $p \in P$. Since it seems unlikely that the firm's equilibrium expected profit is zero, and, moreover, 0 and 1 are weakly dominated by the other prices in the choice set, we can conclude that $\sigma^*(0) = \sigma^*(1) = 0$.

As noted previously, $p = \frac{3}{4}$ is also dominated by $p = \frac{1}{2}$; decreasing the price from $\frac{3}{4}$ to $\frac{1}{2}$ both increases the firm's profit from those who purchase ($\frac{3}{4}\left(1-\frac{3}{4}\right) < \frac{1}{2}\left(1-\frac{1}{2}\right)$) and increases the probability of purchase. So $\sigma^*\left(\frac{3}{4}\right) = 0$.

This means that in any mixed-strategy Nash equilibrium, firms will mix over $\{\frac{1}{4}, \frac{1}{2}\}$.

For simplicity, I only consider the case where $s \geq \frac{5}{32}$. Note that, under this condition, a costly-search consumers who finds price $\frac{1}{2}$ at the first firm he visits will not search. Searching is optimal iff

$$u\left(\frac{1}{2}\right) < \alpha u\left(\frac{1}{4}\right) + (1-\alpha)u\left(\frac{1}{2}\right) - s$$

which is equivalent to $s < \frac{5}{32}\alpha$, which is false, since $s \geq \frac{5}{32} > \frac{5}{32}\alpha$. Therefore, this is an equilibrium in which costly-searchers never search. Thus, firm 1's profit from setting prices $\frac{1}{4}, \frac{1}{2}$, respectively, is

$$\begin{aligned}
\pi_1\left(\frac{1}{4}\right) &= \mu\left[\underbrace{(1-\alpha)\frac{1}{4}\cdot\frac{3}{4}}_{\text{If } p_2=\frac{1}{2}\text{, get all shoppers.}} + \underbrace{\alpha\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{4}}_{\text{If } p_2=\frac{1}{4}\text{, get half shoppers.}}\right] + (1-\mu)\underbrace{\left[\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{4}\right]}_{\text{Get half of costly-searchers.}} \\
&= \frac{3}{32}\left[1+\mu(1-\alpha)\right] \\
\pi_1\left(\frac{1}{2}\right) &= \mu\left[\underbrace{(1-\alpha)\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}}_{\text{If } p_2=\frac{1}{2}\text{, get half shoppers.}} + \underbrace{0}_{\text{If } p_2=\frac{1}{4}\text{, get no shoppers.}}\right] + (1-\mu)\underbrace{\left[\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\right]}_{\text{Get half of costly-searchers.}} \\
&= \frac{1}{8}(1-\alpha\mu)
\end{aligned}$$

Since we must have $\pi_1\left(\frac{1}{4}\right) = \pi_1\left(\frac{1}{2}\right)$, this is a mixed-strategy Nash equilibrium iff $\alpha = \frac{1-3\mu}{\mu}$.

Question 2 (a) If the consumer does not do an additional search, her utility is

$$v - p^* + x - Ns$$

If the consumer does an additional search, there are two possible results. First, the consumer might find that the newly searched product has $\epsilon_{ij} \leq x$. In this case, utility is $v - p^* + x - (N+1)s$. Second, the consumer might find that the newly searched product has $\epsilon_{ij} > x$. In this case, utility is $v - p^* + \epsilon_{ij} - (N+1)s$. So, if the consumer does an additional search, her expected utility is

$$v - p^* + \Pr\left(\epsilon_{ij} \leq x\right)x + \Pr\left(\epsilon_{ij} > x\right)\mathbb{E}\left[\epsilon_{ij} \mid \epsilon_{ij} > x\right] - (N+1)s$$

Since the idiosyncratic term ϵ_{ij} is distributed $U[0,1], \Pr(\epsilon_{ij} < x) = x$ and $\mathbb{E}[\epsilon_{ij} \mid \epsilon_{ij} > x] = \frac{x+1}{2}$. Substituting in and simplifying, expected utility becomes

$$v - p^* + \frac{1 + x^2}{2} - (N+1)s$$

Subtracting the consumer's utility without an additional search, we find that the expected net benefit from an additional search is

$$\frac{1}{2}(1-x)^2 - s \equiv \Delta(x)$$

(b) The consumer searches again if and only if $\Delta(x) \geq 0$, so $\Delta(\underline{x}) = 0$. Inverting Δ gives

$$\underline{x} = 1 - \sqrt{2s}$$

Note that we need to assume $x \leq \frac{1}{2}$, or else the consumer will never search.

(c) A consumer who visits firm j buys from firm j if and only if $u_{ij} \geq \underline{\mathbf{u}}$. Equivalently,

$$\begin{aligned} v - p_j + \epsilon_{ij} &\geq v - p^* + (1 - \sqrt{2s}) \\ \Leftrightarrow \quad \epsilon_{ij} &\geq (1 - \sqrt{2s}) - (p^* - p_j) \end{aligned}$$

Again using the fact that $\epsilon \sim U[0,1]$

$$\Pr \ (\text{buy from } j \mid \ \text{visiting } j) = \sqrt{2s} + (p^* - p_j)$$

(d) To find the equilibrium price p^*, we will first find a firm j 's optimal deviation $\hat{p}$ if all other firms set p^*. Then, the equilibrium is characterized by $\hat{p}(p^*) = p^*$. Firm j 's optimal deviation solves the profit maximization problem

$$\hat{p}(p^*) = \arg\max_{p_j} (p_j - c)\left(\sqrt{2s} + p^* - p_j\right)$$

The first-order condition is

$$\left(\sqrt{2s} + p^* - p_j\right) - (p_j - c) = 0$$

Solving for p_j gives

$$\hat{p}(p^*) = \frac{1}{2}\left(c + \sqrt{2s} + p^*\right)$$

The equilibrium condition $\hat{p}(p^*) = p^*$ then implies that the symmetric pure strategy Nash equilibrium price is $p^* = c + \sqrt{2s}$.

(e) In lecture, we saw a model of differentiated search where there were two firms. We showed that that model had a symmetric pure strategy Nash equilibrium

$$p^* = c + \frac{1}{2 - \sqrt{2s/t}} t$$

The model we considered in this question is like that from lecture, but with $t = 1$ and with infinitely many rather than just two firms. We can show that, for all $s \in \left[0, \frac{1}{2}\right)$,

$$\frac{1}{2 - \sqrt{2s}} > \sqrt{2s}$$

Therefore, the model presented in lecture (with $t = 1$) has higher equilibrium prices that the model in this question. This difference is explained by the number of competing firms. In a model with only two firms, a consumer quickly runs out of firms to search: if she searches both firm 1 and firm 2 and finds both options unfavorable, then she is forced to choose the lesser of these two evils. In a model with infinitely many firms, the consumer always has the option of searching again. The fact that the consumer always has this alternative means that consumers are less "captive," forcing firms to set prices lower.

Question 3

Open ended

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