14.271: Problem Set 8 Solutions

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December 8, 2022

QUESTION 1

(a) Solution: Suppose that both firms mix using a strategy σ with a CDF of F that has full support on $[\underline{v}, \bar{v}]$ with $\underline{v} > c$(otherwise profits are 0).

First, we calculate profits as a function of price, assuming that the other firm prices accoridng to F, which are

$$\Pi(p) = F(p) \cdot (p-c)\frac{\alpha_1}{2} + (1 - F(p))\left(1 - \frac{\alpha_1}{2}\right)(p-c)$$

Next, that $\Pi(\bar{v}) = \frac{\alpha_1}{2}(\bar{v} - c)$ and $\Pi(\underline{v}) = \left(1 - \frac{\alpha_1}{2}\right)(\underline{v} - c)$. As profits should be equalized over all realized prices, this implies that $\underline{v} = \frac{\alpha_1}{2-\alpha_1}(\bar{v} - c) + c$. We also know that $\Pi(p)$ must also be the same as $\Pi(v)$ at all values of p, so

$$F(p) \cdot (p-c)\frac{\alpha_1}{2} + (1 - F(p))\left(1 - \frac{\alpha_1}{2}\right)(p-c) = \frac{\alpha_1}{2}(\bar{v} - c)$$

Solving this for $F(p)$, we find that

$$F(p) = \frac{\alpha_1(\bar{v} - c) - (2 - \alpha_1)(p - c)}{(p-c)(2\alpha_1 - 2)}$$

on $[\underline{v}, \bar{v}]$, with $F(p) = 0$ for $p < \underline{v}$ and $F(p) = 1$ for $p > \bar{v}$.

(b) Consider the case in which both firms price using method A, the case for which both use method B is similar. Without loss of generality, let firm 1 be the one that receives weakly less quantity than firm 2. Consider if firm 1 switched from using method A to method B, but used the same price distribution. Now, they would receive the same expected revenue on the $1 - \alpha_1 - \alpha_2$ group of the population since they behave identically. Further with both the α_1 and α_2 groups now they get $\frac{1}{2}$ of the demand instead of weakly less than $\frac{1}{2}$ the demand.

This generates a strict increase in profits so long as firm 1 is getting strictly less quantity in expectation than firm 2. If firm 1 is getting the same quantity, then note 2 things. 1) There is no pure strategy equilibrium. Hence, I am always mixing along some distribution. Hence, along with moving to method (b) suppose I now always price at the maximum of that distribution. I now have a strictly higher measure of captive consumers, and thus I have a strict incentive to price that way.

QUESTION 2

(a) The optimal price is simply $p = \frac{1}{2}$ coming from maximizing $p(1-p)$. This yields a profit of $\frac{1}{4}$ for any amount of x. Solving for the optimal x is equivalent to maximizing

$$\frac{1}{4} - cx^2$$

which has an optimal of $x = \frac{1}{8c}$ for a profit of $\frac{1}{32c}$. The socially optimal level of advertising equates the marginal benefit to the marginal cost. Here the expected marginal social utility is just $E(v_1)$ which is $\frac{1}{2}$. When equated to marginal cost this implies $x^* = \frac{1}{4c}$. This is lower than the result in Butters. Butters assumes homogeneous consumers

[1] based in part on solutions by Adam Harris, Anton Popov, and Vivek Bhattacharya

which implies the marginal consumer is the same as the expected consumer. However with heterogenous consumers this is not the case and the monopolist will set an advertising level based on the marginal and a social planner will set a level based on the expected consumer. Note: if the firm could engage in perfect price discrimination then they would choose the efficient level.

(b) For a given level of x the proportion of those informed consumers that choose good 1 is given by solving $v_1 - p_1 > 1 - p_2 \iff v_1 > 1 + p_1 - p_2$. Firm 1 is therefore maximizing

$$xp_1(p_2 - p_1)$$

while firm 2 is maximizing

$$(1-x)p_2 + xp_2(1 + p_1 - p_2)$$

this yields the following system of FOC's

$$p_1^* = \frac{p_2}{2}$$
$$p_2^* = \frac{1}{2}(1 + p_1 + \frac{1-x}{x})$$

which solves to

$$p_1^* = \frac{1}{3x}$$
$$p_2^* = \frac{2}{3x}$$

This assumes FOC's are valid and neither firm lies at the end of their demand curve. If $x < \frac{2}{3}$ then p_2^* will remain at 1 and $p_1^* = \frac{1}{2}$

(c) Given these prices firm 1 earns a profit of $\frac{1}{9x}$ given an advertising level of x if $x > \frac{2}{3}$ and a profit of $\frac{x}{4}$ if $x < \frac{2}{3}$. Note that this always yields a solution for x strictly less than $\frac{2}{3}$. The FOC suggests that they will choose $x = \frac{1}{8c}$, if this is greater than $\frac{2}{3}$ then they will just choose $\frac{2}{3}$. Assuming c is sufficiently large then this will give firm 1 a profit of $\frac{1}{32c}$ and firm 2 a profit of $1 - \frac{1}{8c}$

QUESTION 3

(a) In a second-price sealed bid auction in an independent private values setting, it is a dominant strategy for each agent to submit his own valuation as his bid.[2] Then, the environmental group wins iff $v_e > v_\ell$. We know that (v_e, v_ℓ) is distributed uniformly on $[0,2] \times [1,2]$. Viewing this as a rectangle in the (v_e, v_ℓ) plane, we see that the environmental group wins iff the draw lies to the right of the line $v_e = v_\ell$ within this rectangle. This happens with probability 1/4. Thus, the loggers win with probability 3/4.

The expected revenue is the minimum of the two random variables. Note that the probability that the minimum M is less than m is

$$\Pr(M \leq m) = 1 - \Pr(v_e \geq m)\Pr(v_\ell \geq m) = \begin{cases} 1 - \left(1 - \frac{m}{2}\right) = \frac{m}{2} & \text{if } m \in [0,1] \\ 1 - \left(1 - \frac{m}{2}\right)(1 - (m-1)) = -1 + 2m - \frac{m^2}{2} & \text{if } m \in [1,2] \end{cases}.$$

[2] This strategy is weakly dominant in our case. There are of course other equilibria. For instance, we could say b_e is $\alpha \leq 1$ if $v_e \leq 1$, and $b_\ell = v_\ell$. For $\alpha = 1$, we can show through simple probability arguments that the expected revenue is 7/6. (The expected revenue given both types have values above 1 is simply the minimum of two independent random variables distributed uniformly on $[1,2]$, which is 4/3. Otherwise, the revenue is 1. Thus, the expected revenue is the mean, which is 7/6.) If $\alpha = 0$, then the expected revenue is 2/3, by similar reasoning.

To find the expectation of M, we use the tail sum formula. We have

$$\text{expected revenue} = \mathbb{E}[M] = \int_0^\infty \Pr(M \geq m)\ dm = \int_0^1 \left(1 - \frac{m}{2}\right)\ dm + \int_1^2 \left(\frac{m^2}{2} - 2m + 2\right)\ dm = \frac{11}{12}.$$

Revenue equivalence does not hold in our situation. Note from MWG (p. 890) that the condition for the revenue equivalence theorem are (i) risk-neutral bidders, (ii) independent types, (iii) the two mechanisms in question must given bidder i the same probability of getting the good for every realization of types, and (iv) bidder i must have the same utility level in the two mechanisms when his type is at its lowest possible value. There is no *a priori* reason that (iii) or (iv) must hold: it is possible for the logger to shade his bids lower and for a weaker environmentalist to win the good (due to more aggressive bidding), for instance; in the SPA above, the player with the higher valuation always wins.

As mentioned above, therefore there is a tension between the fact that the strong bidder (ℓ) shades his bid downward and the fact that the weak bidder (e) may bid more aggressively, so it is difficult to predict the revenue of the FPA relative to the SPA. We can, however, write down the differential equations we need to solve. The environmentalist maximizes

$$\Pr(\text{win}|b_e)(v - b_e) = \Pr\left((v_\ell < \beta_\ell^{-1}(b_e)\right)(v - b_e) = \left(\beta_\ell^{-1}(b_e) - 1\right)(v - b_e).$$

Similarly, the logger maximizes

$$\frac{\beta_e^{-1}(b_\ell)}{2}\left(v - b_\ell\right).$$

Noting that $v = \beta_\ell^{-1}(b_\ell)$ in this equation, for instance, we have the system

$$\begin{aligned} -\left(\beta_\ell^{-1}(b) - 1\right) + \left(\beta_e^{-1}(b) - b\right)\beta_\ell^{-1\prime}(b) &= 0 \\ -\frac{\beta_e^{-1}(b)}{2} + \left(\beta_\ell^{-1}(b) - b\right)\beta_e^{-1\prime}(b) &= 0, \end{aligned}$$

which is a system of differential equations in the inverse bid functions. Together with the appropriate boundary conditions, we could numerically solve for the bidding behavior and then compute revenues.

(b) There are two very fundamental differences. First, both bidders in this setting know their valuations. Second, valuations are independent. In Hendricks and Porter, there are common values and the informed bidders knows the value exactly whereas the uninformed bidders does not.

(c) It was ok to only discuss the case where reserve price is $R = 0$(as in the problem set), but I keep the solution with R here.

We search for an equilibrium where the logger uses his weakly dominant strategy of just keeping one hand up while $p \leq v_\ell$ and then dropping his hand. We search for the point $p \leq v_e$ at which the environmentalist drops one of his hands. If the logger drops his hand before p (i.e., if $v_\ell \leq p$) then the environmentalist gets both tracts at v_ℓ each and thus earns a profit of $2(v_e - v_\ell)$. If the logger drops his hand after v_ℓ, then the environmentalist gets one tract at a price p and thus earns a profit of $v_e - p$. Thus, the environmentalist's profit, if his valuation is v_e and $p > 1$, is

$$\int_1^p 2(v_e - v_\ell)\ dv_\ell + \int_p^2 (v_e - p)\ dv_\ell = 1 - p(2 - v_e).$$

If $p < 1$, then his profit is $v_e - p$. (Note that this is a clearer manifestation of the intuition that in a multiunit auction you'd want to shade the bid down to influence the price you pay on other units.) This is decreasing in p, so the environmentalist lowers one of his hands immediately (as long as the auction starts at a reserve price $R \leq v_e$, since otherwise he never raises his hands and the logging firm just gets one unit, with the other unit unsold).

We now compute revenue. We break this up into several cases.

- Say $R \leq 1$. Then with probability $R/2$ we have that $v_e \leq R$, in which case only one unit is sold and revenue is R. With probability $1 - R/2$, $v_e \geq R$, so both units are sold and revenue is $2R$. The expected revenue from $R \leq 1$ is $2R - R^2/2$.
- Say $R \geq 1$. Then if $v_e \geq R$ (happens with probability $1 - R/2$) then the revenue is $2R$. If $v_\ell \geq R \geq v_e$ (which happens with probability $(R/2)(2 - R)$), then revenue is R. Otherwise, the revenue is 0. Thus, the revenue is

$$R \cdot \frac{R}{2}(2 - R) + 2R\left(1 - \frac{R}{2}\right) = \frac{1}{2}R(4 - R^2).$$

To compute the probability with which we have the efficient outcome, we have to take a stand on what the reserve price represents. One possibility is that the seller's value is 0 and the point of the reserve is just to affect revenues. Then, the efficient outcome would be for the environmentalist to get both tracts if his valuation exceeds the logger's, and for each to get one tract otherwise. Thus, if $R \leq 1$, then only efficiency is when $v_e \leq R$, which happens with probability $R/2$. If $R \geq 1$, then the possible efficiencies are when $v_e \geq v_\ell \geq R$ (in which case the logger gets one unit when the environmentalist should get two), or $v_\ell \geq R \geq v_e$ (seller only sells one unit), or $R \geq v_\ell, v_e$ (the seller doesn't sell anything at all). This corresponds to a probability of $(R + (2 - R)^2/2)/2$.

Another possibility is that R is the seller's valuation for both tracts. Then, there is no longer an inefficiency if the tract goes unsold (since the reason it was unsold is that the valuation for the players is lower). The inefficiency happens when the environmentalist should get both tracts but the logger gets one since the environmentalist immediately lowers his hand once the logger enters the auction. This is the scenario when $v_e \geq v_\ell \geq R$ and happens with probability $(2 - R)^2/4$ if $R \geq 1$ and $1/4$ if $R \leq 1$.

QUESTION 4

(a) While this problem differs from the standard example of a second-price sealed-bid auction due to its asymmetry, the equilibrium strategies from the standard example are also an equilibrium here: Each player bids her valuation, i.e. $b_i(v_i) = v_i \ \forall i$. Using the standard argument, we can show that a strategy that either shades up or shades down is weakly dominated by bidding one's valuation.

What is the probability that the high-value bidder wins? Let v_H denote the valuation of the high-value bidder and let v_{Li} denote the valuation of a low-value bidder, where the low-value bidders are indexed by i. $v_L^{N:N}$ denotes the largest valuation among the N low-value bidders.

First, note that the CDF of v_{Li} is $\Pr\left(v_{Li} \leq x\right) = \frac{x}{2}$. Then, by independence, $\Pr\left(v_L^{N:N} \leq x\right) = \Pr\left(v_{Li} \leq x \ \forall i\right) = \left(\frac{x}{2}\right)^N$. To get the probability that $v_H > v_L^{N:N}$, we integrate over all possible values of v_H: (Note: The pdf of v_H is $f(x) = 1 \ \forall x \in [1, 2]$.)

$$\begin{aligned}
\Pr\left(v_H > v_L^{N:N}\right) &= \int_1^2 \left(\frac{x}{2}\right)^N dx \\
&= \left(\frac{1}{2}\right)^N \left(\frac{1}{N+1}\right)\left(2^{N+1} - 1\right) \\
&= \frac{2 - 2^{-N}}{N+1}
\end{aligned}$$

(b) For this problem, it is useful to recall (from, for instance, 14.124) the so-called "envelope characterization" of payoffs (also called the ICFOC):

$$V_i(\theta_i) = V_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \bar{y}_i\left(\tilde{\theta}_i\right) d\tilde{\theta}_i \tag{1}$$

That is, if agent i has type θ_i, her expected payoff $V_i(\theta_i)$ is the sum of her expected payoff at her lowest type ($V_i(\underline{\theta}_i)$) and the integral of her probability of being allocated the good $\bar{y}_i\left(\tilde{\theta}_i\right)$ over all types up to θ_i.

In this problem, we know that the expected payoff of an agent i with type θ_i is

$$V_i(\theta_i) = \theta_i \bar{y}_i(\theta_i) - \bar{t}_i(\theta_i) \tag{2}$$

where $\bar{t}_i(\theta_i)$ is her expected transfer to the seller. Combining (1) and (2), we get

$$\bar{t}_i(\theta_i) = \theta_i \bar{y}_i(\theta_i) - V_i(\underline{\theta}_i) - \int_{\underline{\theta}_i}^{\theta_i} \bar{y}_i\left(\tilde{\theta}_i\right) d\tilde{\theta}_i \tag{3}$$

Expected revenue is

$$\begin{aligned}
\mathbb{E}\left[\text{Revenue}\right] &= \sum_i \int_{\Theta_i} \bar{t}_i\left(\tilde{\theta}_i\right) f_i\left(\tilde{\theta}_i\right) d\tilde{\theta}_i \\
&= \int_1^2 \bar{t}_H(\tilde{v}_H) f_H(\tilde{v}_H) d\tilde{v}_H + N \int_0^2 \bar{t}_L(\tilde{v}_L) f_H(\tilde{v}_L) d\tilde{v}_L
\end{aligned} \tag{4}$$

To compute expected revenue in this problem, we will start by using equation (3) to characterize t_H and t_L. Then, we will use equation (4) to compute expected revenue.

We begin with the **high-value** agent. The terms in equation (3) are as follows:

- Recall from (a) that if H has value v_H, her probability of winning the auction is $\bar{y}_H(v_H) = \left(\frac{v_H}{2}\right)^N$.
- $V_H(\underline{\theta}_H) = V_H(1) = \Pr\left(v_L^{N:N} < 1\right)\left(1 - \mathbb{E}\left[v_L^{N:N} \mid v_L^{N:N} < 1\right]\right) = \left(\frac{1}{2}\right)^N \left(1 - \frac{N}{N+1}\right) = \frac{1}{2^N(N+1)}$
- $\int_1^{v_H} \bar{y}_H(\tilde{v}_H) d\tilde{v}_H = \int_1^{v_H} \left(\frac{\tilde{v}_H}{2}\right)^N = \frac{1}{2^N(N+1)}\left(v_H^{N+1} - 1\right)$

Combining these terms,

$$\bar{t}_H(v_H) = \frac{N}{2^N(N+1)} v_H^{N+1}$$

Integrating gives

$$\begin{aligned}
\int_1^2 \bar{t}_H(\tilde{v}_H) f_H(\tilde{v}_H) d\tilde{v}_H &= \int_1^2 \bar{t}_H(\tilde{v}_H) d\tilde{v}_H \\
&= \int_1^2 \frac{N}{2^N(N+1)} \tilde{v}_H^{N+1} d\tilde{v}_H \\
&= \frac{N}{2^N(N+1)(N+2)}\left(2^{N+2} - 1\right)
\end{aligned}$$

Next we consider a **low-value** agent. The terms in equation (3) are as follows:

– The probability of winning for a low-value agent with value v_L is

$$\bar{y}_L\left(v_L\right)=\begin{cases}\left(\frac{v_L}{2}\right)^{N-1}\left(v_L-1\right) & \text{if } v_L\in[1,2]\\ 0 & \text{otherwise}\end{cases}$$

– $V_L\left(\underline{\theta}_L\right)=V_L(0)=0$

– $\int_1^{v_L}\bar{y}_L\left(\tilde{v}_L\right)d\tilde{v}_L=\int_1^{v_L}\left(\frac{\tilde{v}_L}{2}\right)^{N-1}\left(\tilde{v}_L-1\right)=\frac{1}{2^{N-1}}\left[\frac{1}{N+1}\left(v_L^{N+1}-1\right)-\frac{1}{N}\left(v_L^N-1\right)\right]$
$=\frac{1}{2^{N-1}N(N+1)}\left(Nv_L^{N+1}-\left(N+1\right)v_L^N+1\right)$

Combining these terms,

$$\bar{t}_L\left(v_L\right)=\frac{1}{2^{N-1}}\left[\left(\frac{N}{N+1}\right)v_L^{N+1}-\left(\frac{N-1}{N}\right)v_L^N-\frac{1}{N(N+1)}\right]$$

Integrating gives

$$\begin{aligned}\int_1^2\bar{t}_L\left(\tilde{v}_L\right)f_L\left(\tilde{v}_L\right)d\tilde{v}_L&=\int_1^2\frac{1}{2}\bar{t}_L\left(\tilde{v}_L\right)d\tilde{v}_L\\&=\frac{1}{2^N N\left(N+1\right)}\left[\frac{N^2}{N+2}\left(2^{N+2}-1\right)-\left(N-1\right)\left(2^{N+1}-1\right)-1\right]\end{aligned}$$

So, expected revenue is

$$\mathbb{E}\left[\text{Revenue}\right]=\frac{N}{2^N\left(N+1\right)\left(N+2\right)}\left(2^{N+2}-1\right)+\frac{1}{2^N\left(N+1\right)}\left[\frac{N^2}{N+2}\left(2^{N+2}-1\right)-\left(N-1\right)\left(2^{N+1}-1\right)-1\right]$$