Class 2 Outline

1. **Monte-Carlo:**
   - **Framework/Definition**
   - **Algorithm**
   - **Examples**

2. **Random Number Generation**

Monte-Carlo Framework

Estimate $\theta = E[\, h(\mathbf{X}) \,]$

where $\mathbf{X} = \{X_1,\ldots,X_m\}$ is a random vector in R^m,

h(.) is a function $R^m \rightarrow R$,

and $E[\, | h(\mathbf{X}) | \,] < \infty$

Monte-Carlo Algorithm

1. Generate n samples of $\mathbf{X}$: $\mathbf{X}^1, \ldots, \mathbf{X}^n$
2. Compute $h(\mathbf{X}^1), h(\mathbf{X}^2), \ldots, h(\mathbf{X}^n)$
3. Estimate $\theta = E[\, h(\mathbf{X}) \,]$ with

$$\overline{\theta} = [h(\mathbf{X}^1)+h(\mathbf{X}^2)+\ldots+h(\mathbf{X}^n)]/n$$

- Why is $\overline{\theta}$ a good estimator?

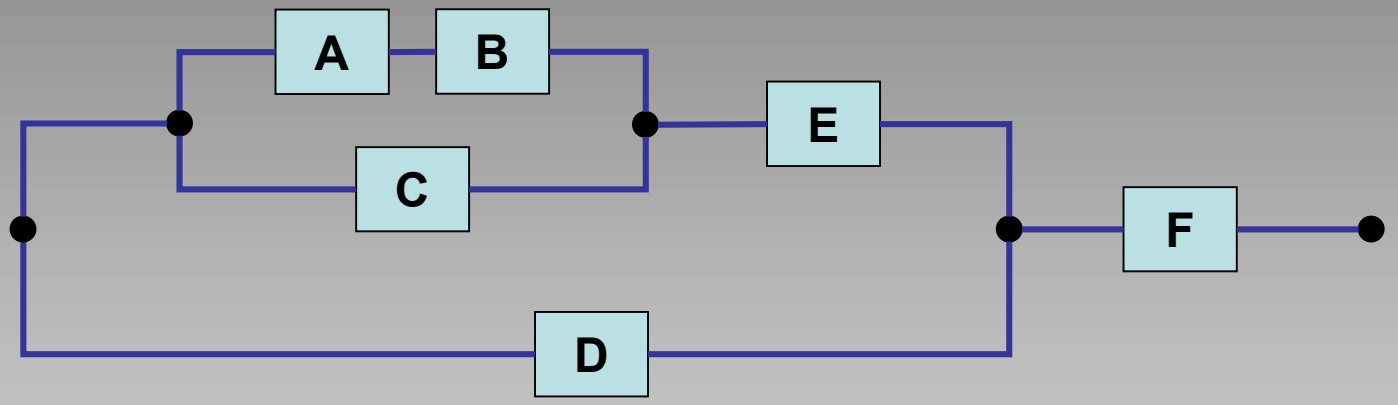
A Sales Incentive Plan

A company has a sales incentive plan with the following structure: In the months when sales are lower than $10,000, a salesrep get a base salary of $2000. If the monthly sales achieved are between $10,000 and $15,000, then he/she is paid 20% of the sales. Finally, if the monthly sales figure is larger than $15,000, the salesrep get 30% of the sales.

Assuming the monthly sales achieved by a salesman follows an exponential distribution with mean $15,000, what is the mean and standard deviation of the salesman's monthly salary under this compensation plan?

Product Reliability

- **Consider a product reliability model given by:**



- **Component life lengths:** $X_A \sim \exp(5.1)$; $X_B \sim N(4.5,1)$; $X_C \sim \exp(6.5)$
  $X_D \sim \exp(5.4)$; $X_E \sim \exp(6.4)$; $X_F \sim N(5.5,0.8)$

- ***What is the probability that the product will function for at least 3 years (warranty period)?***

DreamCast Launch Project

Task	Description	Duration (weeks)	Precedence
A	Market Study	14	
B	External Developer Focus Group	10	
C	Feature Selection	4	A,B
D	Hardware Engineering	25	C
E	Operating System Devlpt.	16	D,B
F	Advertising Campaign	20	D,M,N
G	Supplier Selection & Negotiation	20	D
H	Component Inventory Buildup	45	G
I	Assembly Facility Setup	18	D
J	Finished Good Inventory Buildup	7	I,H
K	Library & Programmer Toolkit Devlpt.	12	E
L	External Development Support Setup	5	K
M	Internal Game Devlpt.	30	K
N	External Game Devlpt.	32	L,O
O	Platform Promotion	6	K
P	Publisher Selection & Negotiation	5	M
Q	Website Setup	20	M
R	Release Promotion Material Design	3	D,M,N
S	Distribution Channels Devlpt. & Negotiation	9	R
T	Carrier Selection & Negotiation	4	S
U	Launch Event Organization & PR	5	R
V	Hardware & Software Shipment	1	J,T,P,N
W	Launch!	0	V,U,Q,F

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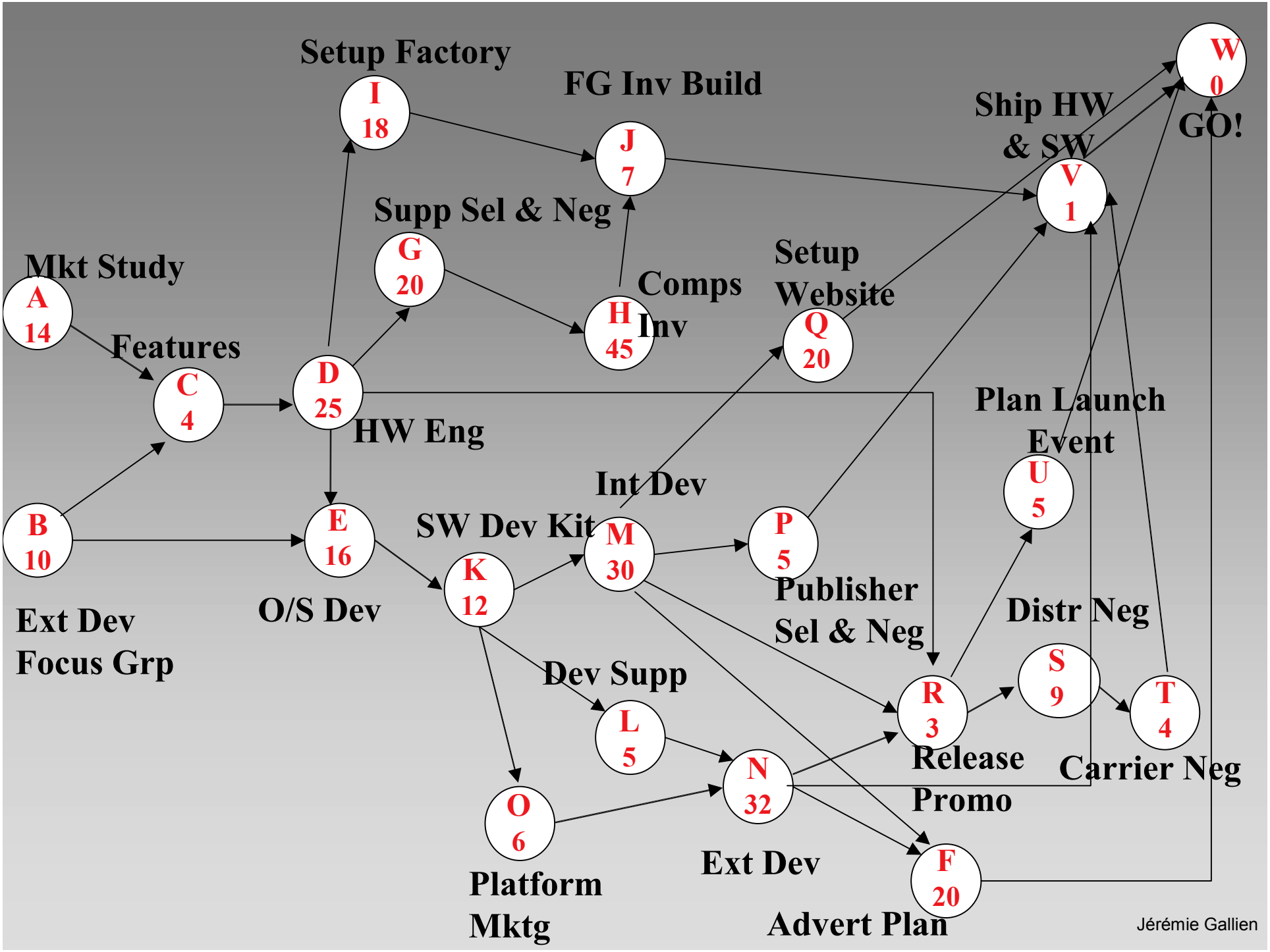
Setup Factory
I
18
FG Inv Build
J
7
W
0
GO!
Ship HW
& SW
V
1
Supp Sel & Neg
G
20
Mkt Study
A
14
Comps
Inv
H
45
Setup
Website
Q
20
Features
C
4
D
25
HW Eng
Plan Launch
Event
U
5
Int Dev
B
10
E
16
SW Dev Kit
M
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P
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K
12
Ext Dev
Focus Grp
O/S Dev
Publisher
Sel & Neg
Distr Neg
S
9
Dev Supp
L
5
R
3
T
4
Release
Promo
Carrier Neg
N
32
O
6
Ext Dev
F
20
Platform
Mktg
Advert Plan
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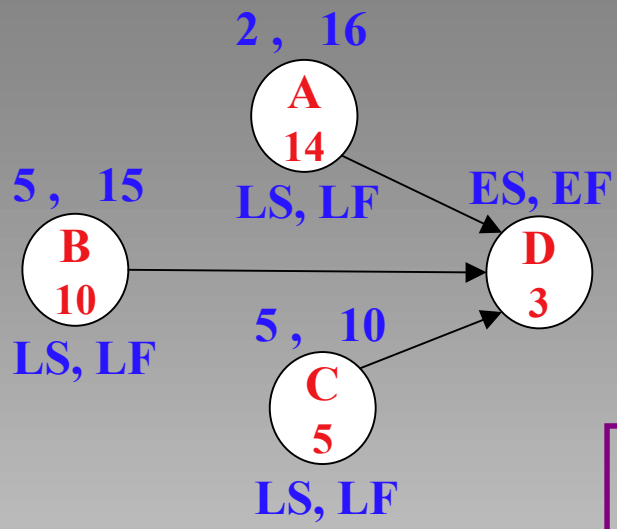
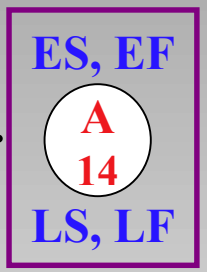
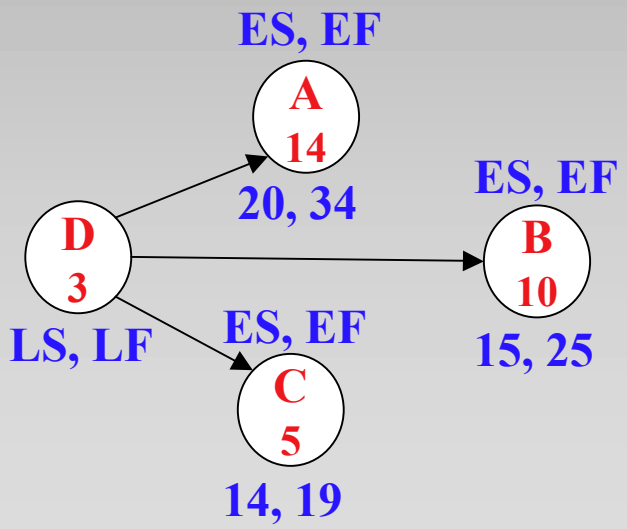
Critical Path Algorithm

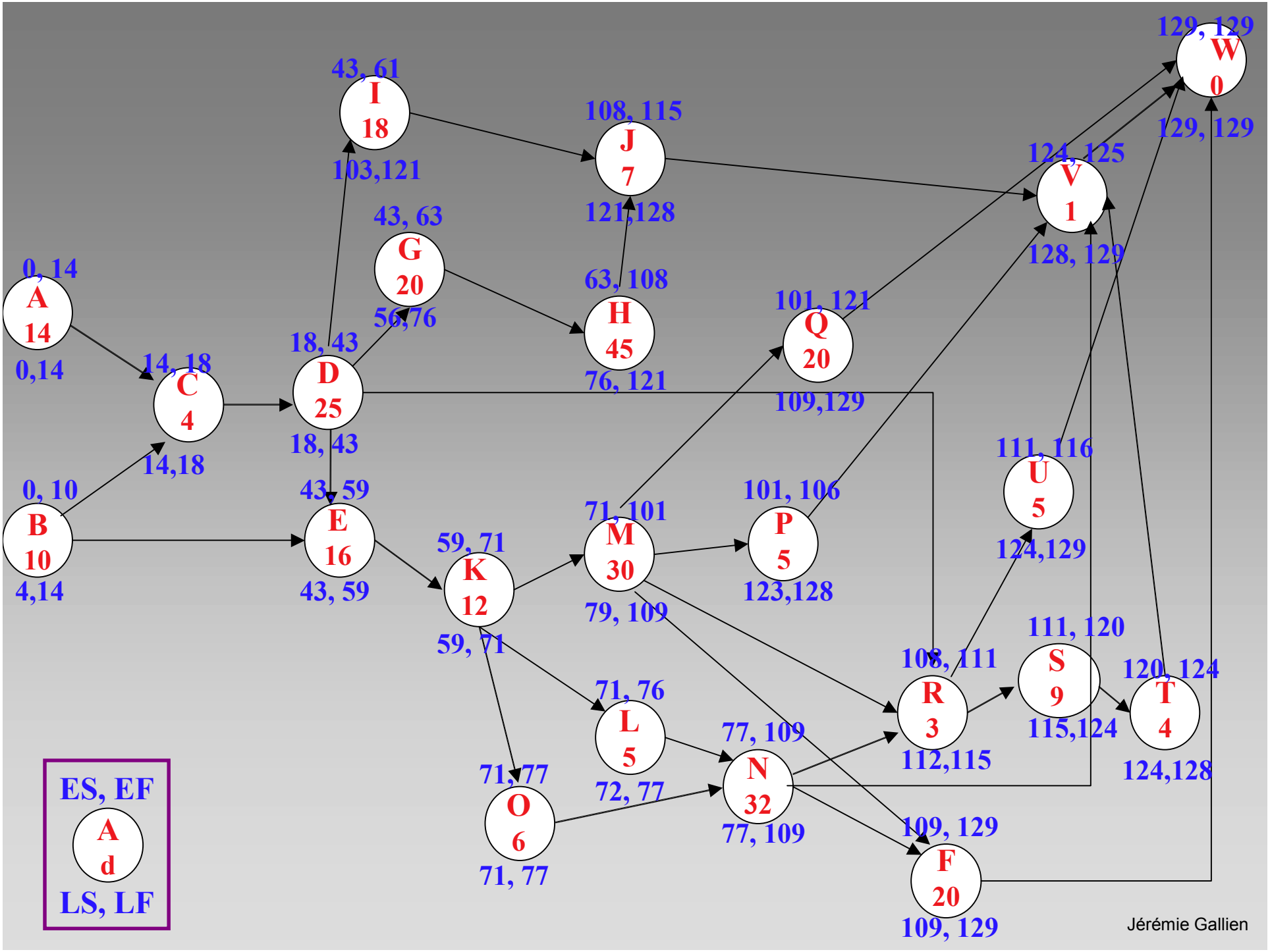
1. A task Early Start Date is the largest of all Early Finish Dates of its predecessors





2. A task Late Finish Date is the earliest of all Late Start Dates of its successors




ES, EF
A
d
LS, LF
A
14
0,14
0, 14
B
10
4,14
0, 10
C
4
14,18
14, 18
D
25
18,43
18, 43
E
16
43, 59
43, 59
G
20
56,76
43, 63
I
18
103,121
43, 61
H
45
76, 121
63, 108
J
7
121,128
108, 115
K
12
59, 71
59, 71
L
5
72, 77
71, 76
M
30
79, 109
71, 101
N
32
77, 109
77, 109
O
6
71, 77
71,77
P
5
123,128
101, 106
Q
20
109,129
101, 121
R
3
112,115
108, 111
S
9
115,124
111, 120
T
4
124,128
120, 124
U
5
124,129
111, 116
V
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128, 129
124, 125
W
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129, 129
129, 129
F
20
109, 129
109, 129
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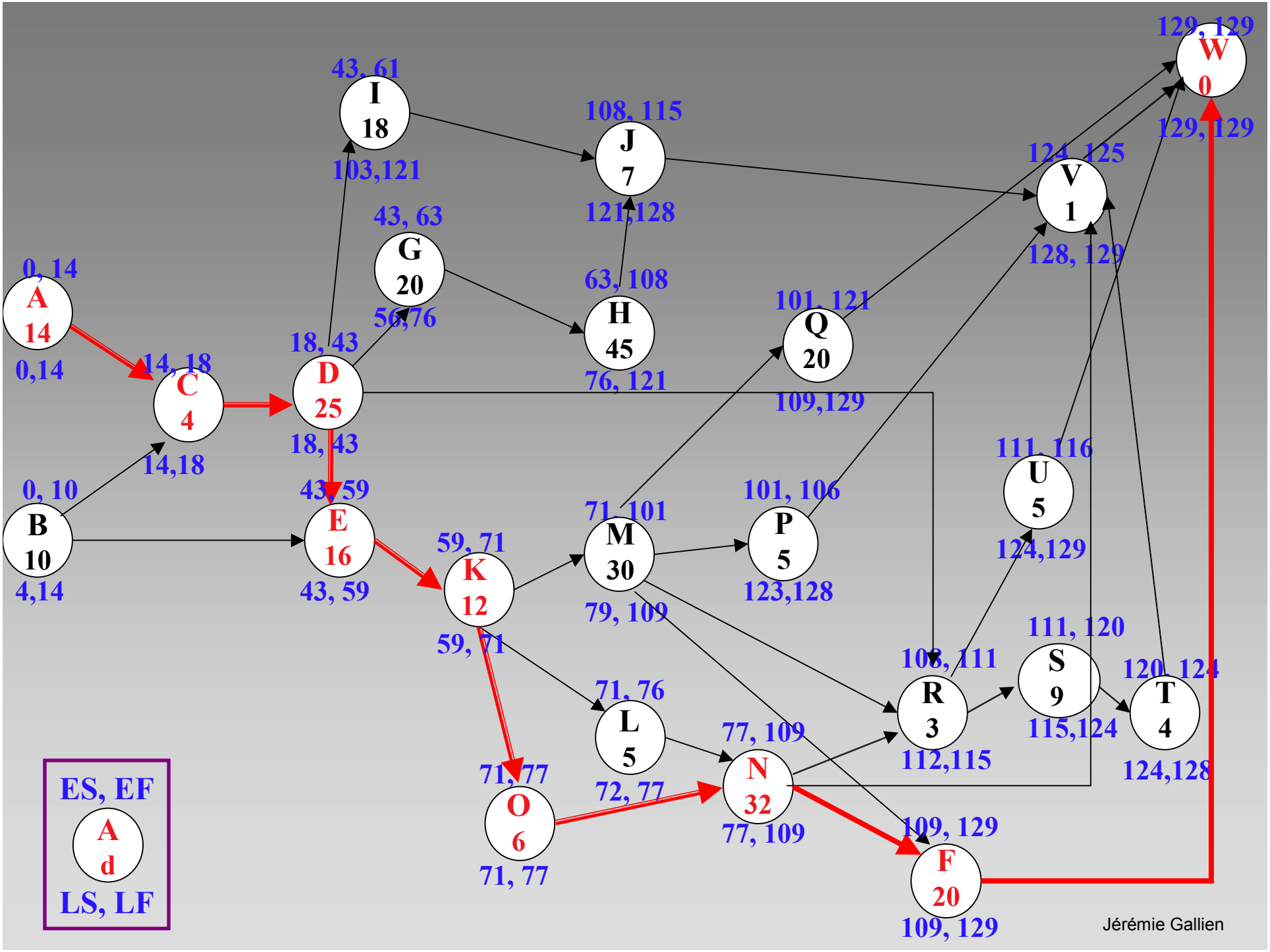
ES, EF
A
d
LS, LF
A 14 0,14 0, 14
B 10 4,14 0, 10
C 4 14,18 14, 18
D 25 18,43 18, 43
E 16 43, 59 43, 59
G 20 56,76 43, 63
I 18 103,121 43, 61
J 7 121,128 108, 115
H 45 76, 121 63, 108
K 12 59, 71 59, 71
L 5 72, 77 71, 76
M 30 79, 109 71, 101
N 32 77, 109 77, 109
O 6 71, 77 71, 77
P 5 123,128 101, 106
Q 20 109,129 101, 121
R 3 112,115 108, 111
S 9 115,124 111, 120
T 4 124,128 120, 124
U 5 124,129 111, 116
V 1 128, 129 124, 125
F 20 109, 129 109, 129
W 0 129, 129 129, 129
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Critical Path Solution

Task	Description	Duration (weeks)	Precedence	LS	LF	ES	EF	Slack
A	Market Study	14		-129	-115	-129	-115	0
B	External Developer Focus Group	10		-125	-115	-129	-119	4
C	Feature Selection	4	A,B	-115	-111	-115	-111	0
D	Hardware Engineering	25	C	-111	-86	-111	-86	0
E	Operating System Devlpt.	16	D,B	-86	-70	-86	-70	0
F	Advertising Campaign	20	D,M,N	-20	0	-20	0	0
G	Supplier Selection & Negotiation	20	D	-73	-53	-86	-66	13
H	Component Engineering & Inventory Buildup	45	G	-53	-8	-66	-21	13
I	Assembly Facility Setup	18	D	-26	-8	-86	-68	60
J	Finished Good Inventory Buildup	7	I,H	-8	-1	-21	-14	13
K	Library & Programmer Toolkit Devlpt.	12	E	-70	-58	-70	-58	0
L	External Development Support Setup	5	K	-57	-52	-58	-53	1
M	Internal Game Devlpt.	30	K	-50	-20	-58	-28	8
N	External Game Devlpt.	32	L,O	-52	-20	-52	-20	0
O	Platform Promotion	6	K	-58	-52	-58	-52	0
P	Publisher Selection & Negotiation	5	M	-6	-1	-28	-23	22
Q	Website Setup	20	M	-20	0	-28	-8	8
R	Release Promotion Material Design	3	D,M,N	-17	-14	-20	-17	3
S	Distribution Channels Devlpt. & Negotiation	9	R	-14	-5	-17	-8	3
T	Carrier Selection & Negotiation	4	S	-5	-1	-8	-4	3
U	Launch Event Organization & PR	5	R	-5	0	-17	-12	12
V	Hardware & Software Shipment	1	J,T,P,N	-1	0	-4	-3	3
W	Launch!	0	V,U,Q,F	0	0	0	0	0

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Stochastic Task Durations

Task	Description	Duration (weeks)	Precedence	LS	LF	ES	EF	Slack
A	Market Study	14		-129	-115	-129	-115	0
B	External Developer Focus Group	10		-125	-115	-129	-119	4
C	Feature Selection	4	A,B	-115	-111	-115	-111	0
D	Hardware Engineering	25	C	-111	-86	-111	-86	0
E	Operating System Devlpt.	16	D,B	-86	-70	-86	-70	0
F	Advertising Campaign	20	D,M,N	-20	0	-20	0	0
G	Supplier Selection & Negotiation	20	D	-73	-53	-86	-66	13
H	Component Engineering & Inventory Buildup	45	G	-53	-8	-66	-21	13
I	Assembly Facility Setup	18	D	-26	-8	-86	-68	60
J	Finished Good Inventory Buildup	7	I,H	-8	-1	-21	-14	13
K	Library & Programmer Toolkit Devlpt.	12	E	-70	-58	-70	-58	0
L	External Development Support Setup	5	K	-57	-52	-58	-53	1
M	Internal Game Devlpt.	30	K	-50	-20	-58	-28	8
N	External Game Devlpt.	32	L,O	-52	-20	-52	-20	0
O	Platform Promotion	6	K	-58	-52	-58	-52	0
P	Publisher Selection & Negotiation							
Q	Website Setup							
R	Release Promotion Material Design							
S	Distribution Channels Devlpt. & Negotiation							
T	Carrier Selection & Negotiation							
U	Launch Event Organization & PR							
V	Hardware & Software Shipment							
W	Launch!							

Duration of development tasks E, M and N now follow exponential distributions with means 16, 30 and 32 respectively.

What is the probability that the total project duration will be more than 135 weeks?

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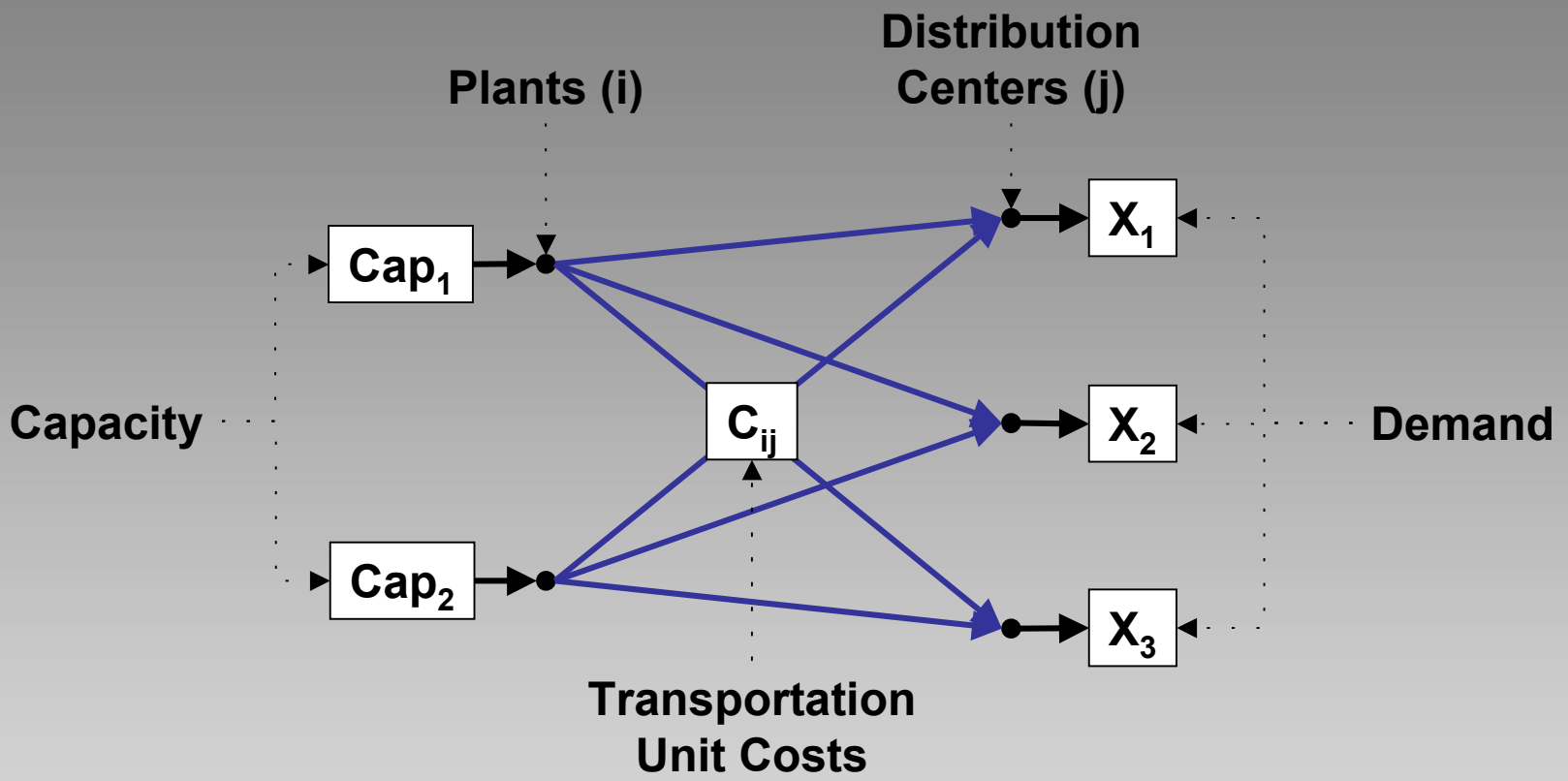
Stochastic Programming



Stochastic Demand

$X_1 \sim N(1500,350)$

$X_2 \sim N(2000,600)$

$X_3 \sim N(1500,450)$

Random Number Generation

- **The key in a Monte-Carlo simulation is to generate sample values drawn from a known probability distribution. How is this done?**

Generate Sample Value from Uniform[0,1]

Discrete Distribution

- **Inverse transform method**
- **Rejection method**

Continuous Distribution

- **Inverse transform method**
- **Rejection method**
- **Composition method**
- **Polar method**

U[0,1] Sample Generation

1. Calculate $Z_1, Z_2, \ldots$ given by the Linear Congruential Generator:

$$Z_i = (aZ_{i-1} + c) \textit{ modulo } m$$

where m, a, c and Z_0 are non-neg. integers

modulus seed

2. Take $U_i = Z_i / m$ as a proxy to an independent sequence of U[0,1] random variates (note: $0 \leq U_i \leq 1$ for all i)

Linear Congruential Generator

- **Example:** $Z_i = (11Z_{i-1})$ *modulo* 16

  $Z_0 = 1$
  $Z_1 = (11) \bmod 16 = 11$
  $Z_2 = (121) \bmod 16 = 9$
  $Z_3 = (99) \bmod 16 = ?$
  $Z_4 = (33) \bmod 16 = ?$

- ***Huge*** **literature on LCG periods, parameter choice, statistical tests for randomness…**
- **Crystal Ball uses** $Z_i = (630{,}360{,}016\ Z_{i-1}) \bmod (2^{31}-1)$

Discrete Distributions

- We want to simulate a random variable X defined by $P(X = x_j) = p_j, \quad j=0,1,\ldots,m$

1. Generate a sample U from U[0,1]
2. if $U < p_0$ then $X = x_0$

   if $p_0 < U < p_0 + p_1$ then $X = x_1$

   .....

   if $p_0 + \ldots + p_{j-1} < U < p_0 + \ldots + p_j$ then $X = x_j$

- Why does this work?

Solution

1. Generate a sample U from U[0,1]
2. if $U < p_0$ then $X = x_0$
   if $p_0 < U < p_0 + p_1$ then $X = x_1$
   …..
   if $p_0 + \ldots + p_{j-1} < U < p_0 + \ldots + p_j$ then $X = x_j$

- $P(X = x_j) = P(p_0 + \ldots + p_{j-1} < U < p_0 + \ldots + p_j) = p_j$

Continuous Distributions

- We want to simulate a random variable X defined by $P(X < t) = F(t)$ (F inversible)

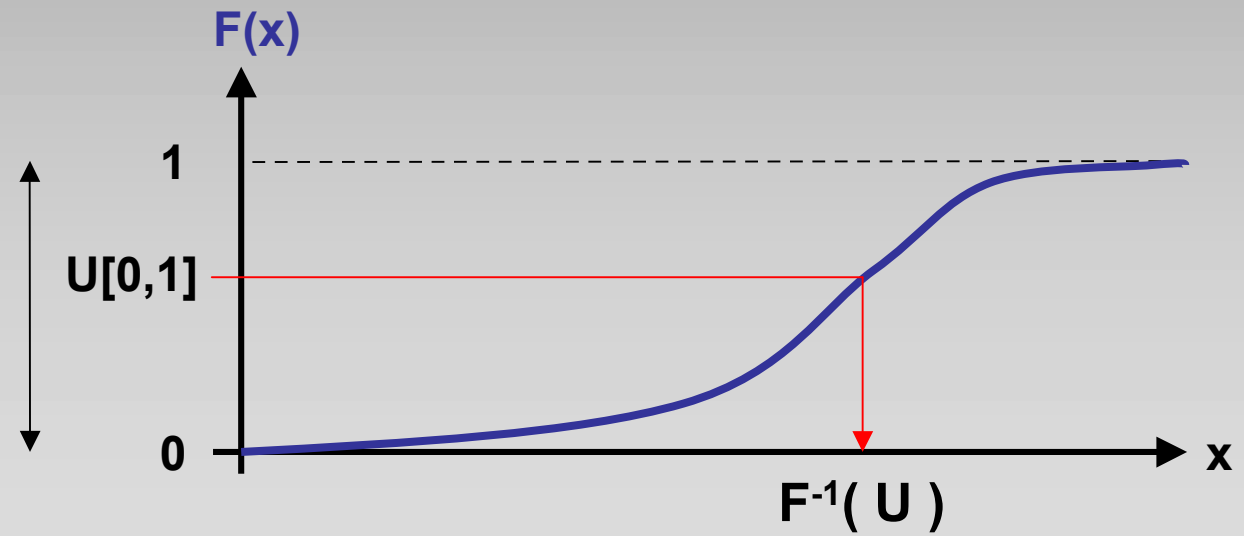
1. Generate a sample U from U[0,1]
2. Set $X = F^{-1}(U)$

- Why does this work?

Solution

1. Generate a sample U from U[0,1]
2. Set $X = F^{-1}(U)$

- $P (X < t) = P(F^{-1}(U) < t) = P(U < F(t)) = F(t)$



Exponential Distribution

- Variable exp(λ) has cdf. $F(t) = 1 - \exp(-\lambda t)$;
  $F^{-1}(y) = - \log(1-y) / \lambda$

1. Generate U from U[0,1]
2. Compute $X = - \log(1-U) / \lambda$
3. The distribution of X is exp(λ) !

Class 2 Wrap-Up

1. Monte-Carlo Framework, Algorithm & Examples
2. Crystal Ball Modeling
3. Theory of Random Number Generation