Overview of Improper Integrals

Now let's contrast the two types of improper integrals we've looked at — one in which x goes to infinity and one in which x approaches a point of singularity.

We have just considered functions like:

$$\frac{1}{x^{1/2}} << \frac{1}{x} << \frac{1}{x^2} \quad \text{as } x \to 0^+.$$

Conversely,

$$\frac{1}{x^{1/2}} >> \frac{1}{x} >> \frac{1}{x^2} \quad \text{as } x \to \infty.$$

In general, we found that improper integrals of functions smaller than $\frac{1}{x}$ converge while improper integrals of functions larger than or equal to $\frac{1}{x}$ diverge. Whether a function is smaller or larger than $\frac{1}{x}$ depends on the function and on what limit we're taking:

$$\frac{1}{x^{1/2}} << \underbrace{\frac{1}{x} << \frac{1}{x^2}}_{\text{integral diverges}} \quad \text{as } x \to 0^+.$$

$$\underbrace{\frac{1}{x^{1/2}} >> \frac{1}{x}}_{\text{integral diverges}} >> \frac{1}{x^2} \quad \text{as } x \to \infty.$$

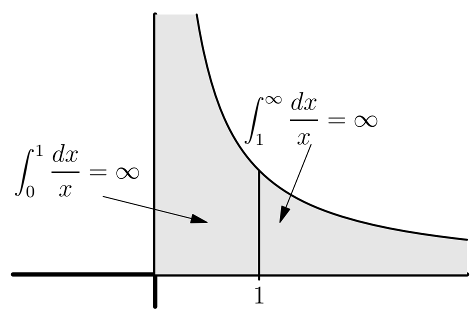

Figure 1: Area under the graph of $y = \frac{1}{x}$.

As shown in Figure 1, the graph of $f(x) = \frac{1}{x}$ is symmetric to itself by a reflection across the line $y = x$. The total area under the curve to the right of $x = 1$ is infinite and so is the area under the curve between $x = 0$ and $x = 1$.

The graph of $y = \frac{1}{x^{1/2}}$ lies below that of $y = \frac{1}{x}$ on the left and above $\frac{1}{x}$ on the right. (See Figure 3.) The total area under the graph of $y = \frac{1}{x^{1/2}}$ to the right of $x = 1$ is infinite, but the area under the curve between $x = 0$ and $x = 1$ is 2. (See Figure 2.)

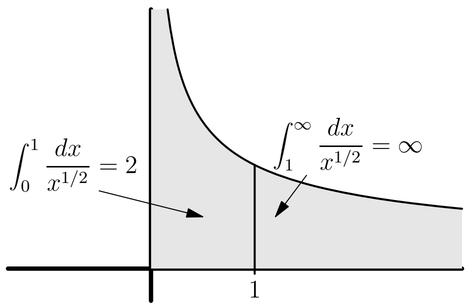

Figure 2: Area under the graph of $y = \frac{1}{x^{1/2}}$.

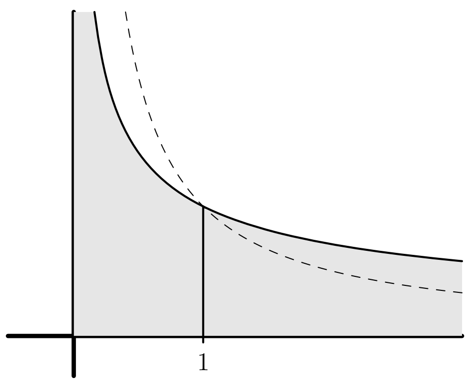

Figure 3: Graph of $y = \frac{1}{x}$ superimposed on graph of $y = \frac{1}{x^{1/2}}$.

Compare this to the area under the graph of $y = \frac{1}{x^2}$. Here the area to the right of 1 is finite (2) and the area between 0 and 1 is infinite. (See Figure 4.)

By comparing the sizes of the vertical and horizontal "tails" of the functions we can get a geometric sense of the difference between convergent and divergent indefinite integrals.

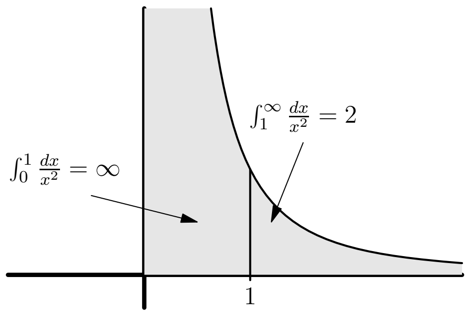

Figure 4: Area under the graph of $y = \frac{1}{x^2}$.

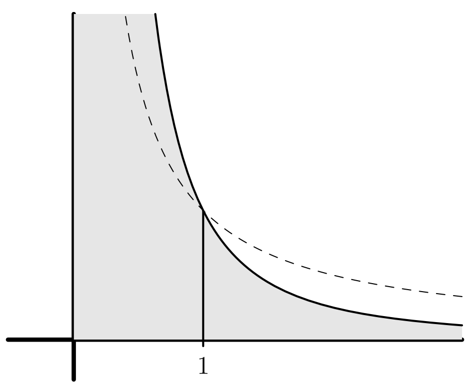

Figure 5: Graph of $y = \frac{1}{x}$ superimposed on graph of $y = \frac{1}{x^2}$.