Arc Length of Parametric Curves

We've talked about the following parametric representation for the circle:

$$\begin{aligned} x &= a\cos t \\ y &= a\sin t \end{aligned}$$

We noted that $x^2 + y^2 = a^2$ and that as t increases the point $(x(t), y(t))$ moves around the circle in the counterclockwise direction.

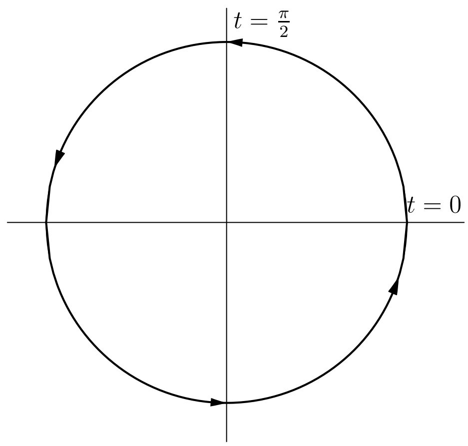

Figure 1: The parametrization $(a\cos t, a\sin t)$ has a counterclockwise trajectory.

We'll now learn how to compute the arc length of the path traced out by this trajectory; the result should match our previous result for the arc length of a circular curve.

Recall our basic relationship:

$$ds^2 = dx^2 + dy^2 \quad \text{or} \quad ds = \sqrt{dx^2 + dy^2}.$$

We incorporate parameter t into this formula as follows:

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

So, to compute the infinitesimal arc length ds we start by computing $\frac{dx}{dt}$ and $\frac{dy}{dt}$:

$$\frac{dx}{dt} = -a\sin t \quad \text{and} \quad \frac{dy}{dt} = a\cos t.$$

Hence,

$$ds = \sqrt{(-a\sin t)^2 + (a\cos t)^2}\, dt$$

$$
\begin{aligned}
&= \sqrt{a^2(\sin^2 t + \cos^2 t)}\, dt \\
&= \sqrt{a^2 \cdot 1}\, dt \\
ds &= a\, dt
\end{aligned}
$$

From this we conclude that the speed at which the point moves around the circle is: $\frac{ds}{dt} = a$. Because the speed is constant, we say that the point is moving with *uniform* speed.

Parametrizations such as:

$$
\begin{aligned}
x &= a \cos kt \\
y &= a \sin kt
\end{aligned}
$$

are common in math and physics classes. Again this is a parametrization of the circle, but this time the point is moving with uniform speed ak. (We'll assume that both a and k are positive.)