

**PROFESSOR:** Welcome back to recitation.

This'll be the last video where we do an optimization problem. And this one's a little bit different than the other two. So I'm going to give you the problem now. The problem is the following-- a cylinder has a fixed volume. What ratio between radius and height minimizes surface area?

Before I give you time to think about that, I'm going to remind you of the two formulas for volume and surface area of a cylinder. So volume of a cylinder is  $\pi r^2 h$ . And surface area is  $2\pi r^2$  plus  $2\pi r h$ .

So with that information, I'll give you some time to work on this problem and then I'll be back.

Welcome back.

OK, what we're doing, again, is we're trying to solve an optimization problem. And so we know now the constraint equation, it's a little different than the other situations because the constraint equation is just we're told that there's a fixed volume but we're not told what it is. That actually will not change how we work on this problem, but it does change the kind of answer I can ask of you. And notice the answer I ask is not about exact values for radius and height, but what ratio between them will minimize surface area. So this is, you'll see at the end, things will look a little different. But ultimately we still have our optimizing equation, which is surface area. And we still have our constraint equation, which is volume.

So again, we're going to do what we always do. We're going to take our optimization equation and use our constraint equation to get rid of a variable so that we can write the right-hand side here in terms of one variable. Let me point out  $V$  is not a variable here.  $V$  is a constant because the volume is fixed. So when you see  $V$ , it's not a variable. It's a constant.

So what do I do? Let me write-- first, let me write surface area in terms of just a function of  $r$ . And I'm going to do something a little tricky which maybe you didn't think of doing, but ultimately you should end up with the same answer once you've simplified. Notice that this term has a  $\pi r^2 h$ . This term also has a  $\pi r^2 h$ . In fact, I can rewrite the volume equation as  $V$  over  $r$  is equal to  $\pi r h$ . So what I'm going to do is take that  $\pi r h$  here and replace it by a  $V$  over  $r$ .

Now again, you might not have done this. You should ultimately get the same answer that I do

when we're finished. And even sooner, probably some simplification would be the same. The only thing I've done here is I've simplified right away. So by looking at the problem and kind of pulling back from the problem I see, oh, there's something here that looks exactly like something over here. So I just want to point that out. That it's not wrong to just substitute for  $h$ , but it's maybe a little faster.

So now the new surface area equation becomes  $2\pi r^2 + \frac{2V}{r}$ . So now we have everything in terms of  $r$ , because again,  $V$  is a constant.

Now let me point out what happens. When  $r$  goes to infinity this term goes to 0, but this term goes to infinity. So as  $r$  gets very large surface area is getting very large. As  $r$  goes to 0 this term goes to 0, but this term goes to infinity.  $V$  is fixed, and when  $r$  goes to 0 this term blows up. So when  $r$  gets as small as we allow or as large as we allow, either way, surface area is going to be getting big. So where this-- where surface area has a derivative with respect to radius it's going to have to be a minimum. So we don't have to check any more-- now we've checked sort of the what's happening towards the boundary at the extreme values of  $r$ .

So now we can, now we can actually solve the problem. Well, what do we do? We're using our optimization equation. We want to take a derivative and set it equal to 0 and find what value for  $r$  gives that. But again, let me point out one more time that in the end I'm asking for a ratio between radius and height. So I'm not going to get all the way to where I have  $r$  equals something. You'll see I'm going to do another trick.

But let me first take the derivative I need. Surface area prime, this is derivative with respect to  $r$ . I have  $2\pi r^2 + \frac{2V}{r}$ . That derivative with respect to  $r$  is  $4\pi r - \frac{2V}{r^2}$ . And this derivative with respect to  $r$ , I'm going to keep the  $2V$ , and the denominator  $1$  over  $r$ , its derivative is negative  $1$  over  $r^2$ . So I'm going to have  $4\pi r$  for the first term and then minus  $\frac{2V}{r^2}$ .

If I set this equal to 0, the derivative equal to 0, and solve, I get to  $\frac{2V}{r^2}$  is equal to  $4\pi r$ , which is the same as-- going to put it on this side--  $r^3$  is equal to  $\frac{2V}{4\pi}$ . Let's just check. If I multiply both sides by  $r^2$ , divide by  $4\pi$ --

oh, I did it backwards. I think it should actually be  $\frac{\pi}{2V}$ . Let me double check that all my signs are correct. That's  $r^3$ , divide by  $4\pi$ , I should get  $\frac{\pi}{2V}$ . I should not. I've been told from the audience I made a mistake. Sometimes you'll see when you're at the board and on video, scary things can happen.  $2V$ -- oh. Multiply through by  $r$ , I get  $r^3$ . Oh, I have the  $2V$  in the numerator. I apologize. The  $2V$  is in the numerator, the  $4\pi$  is in the denominator. So

I get  $V$  over  $2\pi$ . Does that look better, audience? The audience tells me that looks better.

OK. So here I am. I have  $r$  cubed is equal to  $V$  over  $2\pi$ . Now, if I wanted to I could take the cube root of both sides and get  $r$  explicitly in terms of  $V$  and  $2\pi$ .  $V$  is a constant,  $\pi$  is a constant, I would be done. But I didn't ask for what  $r$  actually is. I asked for a ratio. So let's make this problem simpler.

All I need in the end is  $r$  divided by  $h$ . Let's go back to a formula we have and see if we can figure out a way to get  $r$  divided by  $h$ . Look at this formula for  $V$ . It has an  $r$  squared and it has an  $h$ . So if I divide by  $r$  squared  $h$  on this side, I'll end up with an  $r$  over  $h$ . Hopefully you buy that. I'm going to even write it out. I'm going to take  $r$  cubed and divide it by  $r$  squared  $h$ . That in the end, is  $r$  over  $h$ . Let's go back here look at what that equals.  $r$  squared  $h$  is equal to  $V$  over  $\pi$ . Right?  $r$  squared  $h$  is equal to  $V$  over  $\pi$ . So I can divide this side by  $r$  squared  $h$  and divide this side by  $V$  over  $\pi$  and it's the same thing. Which is, by the way, the same thing as multiplying by  $\pi$  over  $V$ . Right?

So I can multiply this side by  $\pi$  over  $V$ , and what do I get? I get the  $V$ 's divide out, the  $\pi$ 's divided by-- divide out, and I get  $1/2$ . So the result is that the ratio between radius and height should be 1 to 2.

Let me one more time explain what we were doing here. In the end, the answer, the question just asks, what is the ratio between radius and height that minimizes surface area? And I had an  $r$  cubed, I wanted to get  $r$  divided by  $h$ . So I divided by two  $r$ 's-- I divided by  $r$  squared-- and then  $h$  again. But  $r$  squared  $h$  is  $V$  over  $\pi$ , so if I divide by  $r$  squared  $h$  I'm dividing by  $V$  over  $\pi$ . So I can do on the right-hand side the same thing that I do on the left-- I divided by  $V$  over  $\pi$ , which is multiplying by  $\pi$  over  $V$ . The  $\pi$ 's divide out, the  $V$ 's divide out, and I'm left with  $1/2$ .

And so this was an optimization problem where the constraint equation did not have a number in it, but it did have, it did have a fixed constant. So I couldn't ask you for an exact value for the radius and an exact value for the height, but I could ask you for how they relate. And that's ultimately what I did.

And I think that's where we'll stop.