Using Simpson's Rule for the normal distribution

This problem uses Simpson's rule to approximate a definite integral important in probability.

In our probability unit, we learned that when given a probability density function $f(x)$, we may compute the probability P that an event x is between a and b by calculating the definite integral:

$$P(a \leq x \leq b) = \int_a^b f(x)\,dx.$$

Here we're assuming that a probability density function $f(x)$ has the property that

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

In the next session, we will show that $f(x) = \frac{1}{\sqrt{\pi}}e^{-x^2}$ is a probability density function with this property. For now, we assume this property.

Question: Suppose the probability density function for American male height is roughly (in inches x)

$$h(x) = \frac{1}{2.8\sqrt{2\pi}}e^{-(x-69)^2/5.6}.$$

- Use Simpson's rule to estimate the probability that an American male is between 5 and 6 feet tall.
- Use Simpson's rule to estimate the probability that an American male is over 8 feet tall.

Solution:

For the first part, 5 feet = 60 inches and 6 feet = 72 inches, so we must compute

$$P(60 \leq x \leq 72) = \int_{60}^{72} h(x)\,dx = \int_{60}^{72} \frac{1}{2.8\sqrt{2\pi}}e^{-(x-69)^2/5.6}\,dx$$

(**NOTE:** We've seen that e^{-x^2} has no elementary antiderivative, so we can't just compute the definite integral using the Fundamental Theorem of Calculus. Some numerical integration is required.)

We can make a table of values of $h(x)$ using a calculator (rounded to three decimal places):

x	$h(x)$
60	$7.45 * 10^{-8}$
62	$2.26 * 10^{-5}$
64	$1.60 * 10^{-3}$
66	$2.86 * 10^{-2}$
68	0.119
70	0.119
72	$2.86 * 10^{-2}$

Now using Simpson's rule, we estimate the definite integral to be:

$$\frac{\Delta x}{3}\left(h(60) + 4h(62) + 2h(64) + 4h(66) + 2h(68) + 4h(70) + h(72)\right)$$

where our width Δx of each interval is 2 inches. This is approximately .574 (or 57.4 percent).

For the second part, we need to make some assumptions about the normal distribution since, strictly speaking, the probability would be given by

$$P(8 < x < \infty) = \int_8^\infty h(x)\,dx.$$

We don't expect many people to be over 8 feet 4 inches. Indeed, $h(100)$ is *extremely small* and h is decreasing. So we may estimate the above integral to high accuracy using the definite integral from $x = 96$ to $x = 100$, which in turn may be estimated by Simpson's rule. Making a similar table:

x	$h(x)$
96	$4.15 * 10^{-58}$
98	$8.55 * 10^{-67}$
100	$4.22 * 10^{-76}$

Then Simpson's rule estimates the integral:

$$\int_{96}^{100} h(x)\,dx = \frac{2}{3}\,(h(96) + 4h(98) + h(100)) = 2.77 * 10^{-58}$$

This is a very small number, and so even though there are over 300 million Americans, of which roughly half are male, we expect essentially no chance of seeing a person over 8 feet tall based on our model using the normal distribution.