Taylor's Formula

Taylor's formula describes how to get power series representations of functions. The function e^x doesn't look like a polynomial; we have to figure out what the values of a_i have to be in order to describe e^x as a series.

Taylor's formula says that given any function f for which the n^{th} derivative $f^{(n)}(x)$ exists for x near 0,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

We'll learn how to use it soon.

Why should this work? Suppose that:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

Then:

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots \quad \text{and}$$
$$f''(x) = 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + \cdots \quad \text{and}$$
$$f^{(3)}(x) = 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4 x + \cdots$$

Evaluating each of these at 0 we see that: $f(0) = a_0$, $f'(0) = a_1$, $f''(0) = 2a_2$ and $f^{(3)}(0) = 3 \cdot 2a_3$. Solving for a_3 we get $a_3 = \dfrac{f^{(3)}(0)}{3 \cdot 2 \cdot 1}$ and in general:

$$a_n = \frac{f^{(n)}(0)}{n!},$$

where:

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 1.$$

We define $0! = 1$ because that makes our formulas work nicely.