Another Moving Exponent

Find the value of:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Technically this is not a calculus problem, but we will use some calculus to solve it. There are two reasons to discuss this now – first that the answer is very interesting and second that it has another moving exponent – the exponent n in the problem is changing as we take the limit.

Whenever we're faced with a moving exponent our first step is to use a logarithm to turn the exponent into a multiple:

$$\ln\left[\left(1+\frac{1}{n}\right)^n\right] = n\ln\left(1+\frac{1}{n}\right).$$

Now we want to start thinking about the limit of this quantity as n approaches infinity. We've had a lot of practice thinking about limits as Δx approaches zero and very little practice with numbers approaching infinity, so it makes sense to try to rephrase this from a question about a very large number n to a question about a very small number Δx.

The quantity $\Delta x = 1/n$ will approach zero as n goes to infinity. If $\Delta x = 1/n$ then $n = 1/\Delta x$, and we get:

$$\lim_{n\to\infty}\left[n\ln\left(1+\frac{1}{n}\right)\right] = \lim_{\Delta x\to 0}\left[\frac{1}{\Delta x}\ln(1+\Delta x)\right].$$

This doesn't look like much of an improvement, but by subtracting $0 = \ln 1$ from $\ln(1+\Delta x)$ we can put it in a familiar form:

$$\begin{aligned}
\lim_{\Delta x\to 0}\left[\frac{1}{\Delta x}\ln(1+\Delta x)\right] &= \lim_{\Delta x\to 0}\left[\frac{1}{\Delta x}\left(\ln(1+\Delta x)-\ln 1\right)\right] \\
&= \lim_{\Delta x\to 0}\frac{\ln(1+\Delta x)-\ln 1}{\Delta x} \\
&= \left.\frac{d}{dx}\ln x\right|_{x=1} \\
&= \left.\frac{1}{x}\right|_{x=1} \\
&= 1
\end{aligned}$$

By strategically subtracting zero ($\ln 1$), we were able to turn this ugly limit into a difference equation, which we could then evaluate using calculus.

Now we just have to work backward to figure out the answer to our original question.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} e^{\ln\left[\left(1+\frac{1}{n}\right)^n\right]}$$

$$
\begin{aligned}
&= e^{\lim_{n\to\infty} \ln\left[\left(1+\frac{1}{n}\right)^n\right]} \\
&= e^1 \\
&= e
\end{aligned}
$$

That's right,

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

and we now have a way to get a numerical value for e. Using this formula we can find the value of e with as much precision as our calculators will allow. For example,

$$\left(1+\frac{1}{10000}\right)^{10000} \cong 2.7182.$$