Maximum Area of Two Squares

Consider a wire of length 1, cut into two pieces. Bend each piece into a square. We want to figure out where to cut the wire in order to enclose as much area in the two squares as possible.

In all of these problems you start with a "bunch of words" — a story problem. The two main tasks in starting the problem are to draw a diagram and to pick variables.

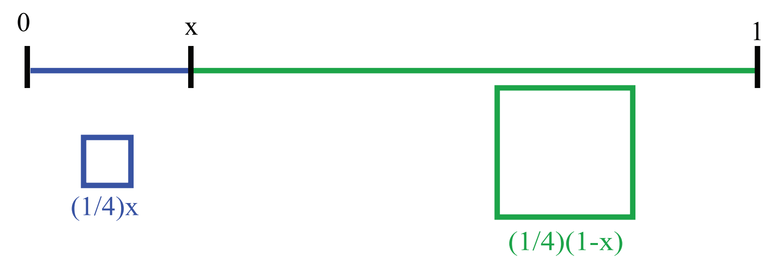

Figure 1: Two pieces of wire enclose two squares.

If we cut the wire so that one piece has length x, the other piece will have length $1 - x$. We know that we're bending these pieces of wire into squares, so we can add those squares to our diagram. The first square will have sides of length $\frac{x}{4}$ and the second square will have sides of length $\frac{1-x}{4}$.

We want to find a maximum area, so we'll need formulas for the areas of these squares. The first square's area is $\frac{x^2}{16}$ and the second square has area $\left(\frac{1-x}{4}\right)^2$. The total area is then

$$\begin{aligned} A &= \left(\frac{x}{4}\right)^2 + \left(\frac{1-x}{4}\right)^2 \\ &= \frac{x^2}{16} + \frac{(1-x)^2}{16} \end{aligned}$$

Most calculus students' instinct at this point is to find the critical points — the values x_0 for which $A'(x_0) = 0$. We can do that now if you like:

$$\begin{aligned} A' &= \frac{2x}{16} + \frac{2(1-x)}{16}(-1) \\ &= \frac{x}{8} - \frac{1}{8} + \frac{x}{8} \\ A' &= \frac{2x-1}{8} \\ A' &= 0 \implies 2x - 1 = 0 \implies x = \frac{1}{2} \end{aligned}$$

So $x = \frac{1}{2}$ is a critical point with critical value:

$$A\left(\frac{1}{2}\right) = \left(\frac{\frac{1}{2}}{4}\right)^2 + \left(\frac{\frac{1}{2}}{4}\right)^2 = \frac{1}{32}$$

We're not done yet, though. We still need to check the endpoints! The length x of the first piece of wire has to satisfy $0 < x < 1$, so we should check the limits as x approaches the endpoints. In this case we can find those limits just by plugging in values. At $x = 0$,

$$A(0^+) = 0^2 + \left(\frac{1-0}{4}\right)^2 = \frac{1}{16}.$$

At $x = 1$,

$$A(1^-) = \left(\frac{1}{4}\right)^2 + 0^2 = \frac{1}{16}.$$

If we try to graph the function with the information we have now, we see that it starts at the point $\left(0, \frac{1}{16}\right)$, dips down to the point $\left(\frac{1}{2}, \frac{1}{32}\right)$, then goes back up to $\left(1, \frac{1}{16}\right)$. (See Fig. **??**.)

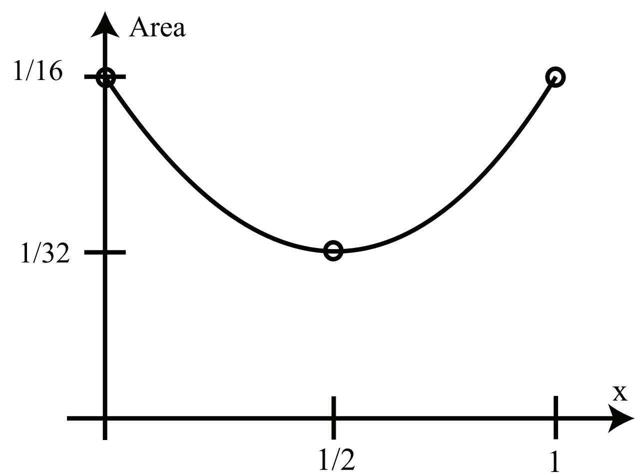

Figure 2: Graph of the area function.

When we found the critical point we did not find the maximum enclosed area — the *minimum* area was achieved at $x = \frac{1}{2}$. The maximum area is not achieved in $0 < x < 1$, but it is achieved at $x = 0$ or 1. The maximum corresponds to using the whole length of wire for one square.

Moral: If you don't pay attention to what the function looks like you may find the worst answer, rather than the best one.

We conclude that the least area enclosed by the two squares is $\frac{1}{32}$, when $x = \frac{1}{2}$; i.e when the two squares are equal. The greatest area enclosed is $\frac{1}{16}$ when $x = 0$ or $x = 1$ and there is only one square.

Commonly asked questions:

What is the minimum? It's the minimum value $\frac{1}{32}$.

Where is the minimum? It's at the critical point $x = \frac{1}{2}$.

These are two different questions. Be sure to answer the correct one; you may get so involved in doing the calculus to find $\frac{1}{2}$ that you forget to find the minimum value $\frac{1}{32}$. Both the critical point and the critical value are important; together they form the point on the graph $(\frac{1}{2}, \frac{1}{32})$ where it turns around.

There are many more and less precise ways to ask these questions. You'll have to do your best to understand what the questions (and answers) mean from the context of the question or example.

Question: Since the goal was to enclose as much area as possible, why did we find the minimum area?

Answer: The reason is that when we go about our procedure of looking for the least or the most, we'll automatically find both. We won't know which one is which until we compare values. It's actually to your advantage to figure out both the maximum and minimum whenever you answer such a question; otherwise you won't understand the behavior of the function very well.

Question: Could we also use the second derivative test here?

Answer: Yes, and we're going to see an example of the second derivative test soon. We could also look at the equation of $A(x)$ and notice that the graph must be a parabola that opens upward.