Power Series Expansion of the Error Function

Several times in this course we've seen the error function:

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt.$$

(The factor of $\frac{2}{\sqrt{\pi}}$ guarantees that $\lim_{x\to\infty} \text{erf}(x) = 1$.) This function is very important in probability theory, but we don't have a conventional algebraic description of it.

Because we can integrate power series and know that:

$$e^{-t^2} = 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots \quad (R = \infty),$$

we can now find a power series expansion for the error function.

$$\begin{aligned}
\text{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt \\
&= \frac{2}{\sqrt{\pi}} \int_0^x (1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots)dt \\
&= \frac{2}{\sqrt{\pi}} \left[t - \frac{t^3}{3} + \frac{t^5}{5\cdot 2!} - \frac{t^7}{7\cdot 3!} + \cdots \right]_0^x \\
&= \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots \right)
\end{aligned}$$

To get this to look exactly like a power series we would distribute the factor of $\frac{2}{\sqrt{\pi}}$ across the sum, multiplying it by each term of the series. However, that's not strictly necessary.

This turns out to be a very good way to compute the value of the error function; your calculator probably uses this method.