

When does a function equal its Taylor series?

We have computed the Taylor series for a differentiable function, and earlier in the course, we explored how to use their partial sums, i.e. Taylor polynomials, to approximate the function. So we know that the Taylor series can be quite useful. But we haven't addressed the question of when a function is equal to its Taylor series. We do this now.

Recall that when we write down an infinite series with upper bound " ∞ ," we mean the following:

$$\sum_{n=0}^{\infty} a_n = \lim_{N \rightarrow \infty} \sum_{n=0}^N a_n$$

In the context of Taylor series for a function f , this means that the Taylor series $T_f(x)$ is expressible as a limit of Taylor polynomials $P_N(x)$ as $N \rightarrow \infty$. That is, the notation

$$T_f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n \quad \text{means} \quad T_f(x) = \lim_{N \rightarrow \infty} \sum_{n=0}^N \frac{f^{(n)}(c)}{n!} (x-c)^n = \lim_{N \rightarrow \infty} P_N(x),$$

where $P_N(x)$ is the N -th degree Taylor polynomial to $f(x)$ at $x = c$.

Hence $f(x)$ will equal $T(x)$ provided

$$f(x) - \lim_{N \rightarrow \infty} P_N(x) = \lim_{N \rightarrow \infty} (f(x) - P_N(x)) = 0.$$

For each N , the function $(f(x) - P_N(x))$ is often called the "remainder" (i.e., what remains after taking the N -th Taylor polynomial from f) and denoted $R_N(x)$. Thus we must find all x for which

$$\lim_{N \rightarrow \infty} R_N(x) = 0.$$

A very useful bound on $R_N(x)$ is given by Taylor's inequality:

Taylor's Inequality. *If $|f^{(N+1)}(x)| \leq B$ for all x in the interval $[c-d, c+d]$, then the remainder $R_N(x)$ (for the Taylor polynomial to $f(x)$ at $x = c$) satisfies the inequality*

$$|R_N(x)| \leq \frac{B}{(N+1)!} |x-c|^{N+1} \quad \text{for all } x \text{ in } [c-d, c+d].$$

If the right-hand side of Taylor's inequality goes to 0 as $N \rightarrow \infty$, then the remainder must go to 0 as well, and hence for those x values, the function matches its Taylor series.

Question: Use Taylor's inequality to show that e^x converges to its Taylor series at 0 for all real x .

Solution:

We jump straight in and use Taylor's inequality to try to show that the right-hand side of the inequality goes to 0 when we take the limit as $N \rightarrow \infty$. The $(n+1)$ -th derivative of e^x is just e^x , which is an increasing function. So on any interval $[-d, d]$, we immediately obtain the bound:

$$|f^{(N+1)}(x)| \leq e^d \quad \text{for all } x \text{ in } [-d, d]$$

which is, in particular, independent of N . Now taking limits of both sides of Taylor's inequality gives:

$$\lim_{N \rightarrow \infty} |R_N(x)| \leq \lim_{N \rightarrow \infty} \frac{e^d}{(N+1)!} |x|^{N+1} \quad \text{for all } x \text{ in } [-d, d].$$

Now analyzing the right-hand side of this inequality,

$$\lim_{N \rightarrow \infty} \frac{e^d}{(N+1)!} |x|^{N+1} = e^d \lim_{N \rightarrow \infty} \frac{|x|^{N+1}}{(N+1)!} \leq e^d \lim_{N \rightarrow \infty} \frac{d^{N+1}}{(N+1)!} = 0 \quad \text{for all } x \text{ in } [-d, d].$$

because $(N+1)!$ grows much faster than d^{N+1} .

We've now shown that given *ANY* positive number d , $R_N(x) \rightarrow 0$ as $N \rightarrow \infty$ for all x in $[-d, d]$. Since d was arbitrary, we see that $R_N(x) \rightarrow 0$ for any real x , and hence e^x is equal to its Taylor series at 0 for all x . That is,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

In particular, we have a rather elegant description of e as

$$e = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$

which can be used to calculate approximations to e which are accurate to arbitrarily many decimal places.

These are difficult questions, but now that you've seen the worked example for e^x , try to do the same for the Taylor series of either $\sin(x)$ or $\cos(x)$ at 0. (Hint: To bound the size of the derivatives of sine and cosine, use the fact that their range is $[-1, 1]$.)

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