Area of an Off Center Circle

Let's find the area in polar coordinates of the region enclosed by the curve $r = 2a\cos\theta$. We've previously shown that this curve describes a circle with radius a centered at $(a, 0)$. In rectangular coordinates its equation is $(x-a)^2 + y^2 = a^2$.

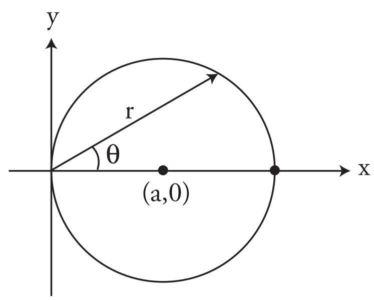

Figure 1: Off center circle $r = 2a\cos\theta$.

We're going to integrate an infintessimal amount of area dA. The integral will go from $\theta_1 = -\frac{\pi}{2}$ to $\theta_2 = \frac{\pi}{2}$. We could find these limits by looking at Figure 1; to draw the circle we might start by moving "down" at angle $-\frac{\pi}{2}$. As we move along the bottom of the circle toward $(2a, 0)$ the angle increases to 0, and as we trace out the top of the circle we're moving from angle 0 to angle $\frac{\pi}{2}$ ("up").

We might also find the limits of integration by looking at the formula and realizing that the cosine function is positive for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. When $\theta = \pm\frac{\pi}{2}$, $r = 2a\cos\theta$ is 0, so the two ends of the curve meet at the origin.

Our infinitessimal unit of area is $dA = \frac{1}{2}r^2\,d\theta$, so:

$$\begin{aligned}
A &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}r^2\,d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(2a\cos\theta)^2\,d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2a^2\cos^2\theta\,d\theta \\
&= 2a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta \qquad \text{(half angle formula)} \\
&= 2a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\,d\theta \\
&= a^2\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}
\end{aligned}$$

$$
\begin{aligned}
&= a^2\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right) \\
&= \pi a^2.
\end{aligned}
$$

We know that the area of a circle of radius a is πa^2; our answer is correct.