Implicit Differentiation Example

How would we find $y' = \frac{dy}{dx}$ if $y^4 + xy^2 - 2 = 0$?

We could use a trick to solve this explicitly — think of the above equation as a quadratic equation in the variable y^2 then apply the quadratic formula:

$$\begin{aligned} y^2 &= \frac{-x \pm \sqrt{x^2+8}}{2}, \\ &\text{so} \\ y &= \pm\sqrt{\frac{-x \pm \sqrt{x^2+8}}{2}}. \end{aligned}$$

Since we see $\pm$ twice in this equation, there are four possible branches to consider. This means that to be thorough we'd want to compute four different derivatives. This is a lot of work.

Instead, we can compute $\frac{dy}{dx}$ using implicit differentiation. As always, we start by applying $\frac{d}{dx}$ to both sides:

$$\begin{aligned} \frac{d}{dx}(y^4 + xy^2 - 2) &= \frac{d}{dx}0 \\ \frac{d}{dx}(y^4) + \frac{d}{dx}(xy^2) - \frac{d}{dx}2 &= 0 \\ 4y^3\frac{dy}{dx} + (y^2 + x \cdot 2y\frac{dy}{dx}) - 0 &= 0 \\ 4y^3\frac{dy}{dx} + 2xy\frac{dy}{dx} &= -y^2 \\ (4y^3 + 2xy)\frac{dy}{dx} &= -y^2 \\ \frac{dy}{dx} &= \frac{-y^2}{4y^3 + 2xy} \end{aligned}$$

In lecture Professor Jerison used the shorthand y' for the derivative; here we use $\frac{dy}{dx}$ to make it clear that we are differentiating with respect to x.