Implicit Differentiation and Min/Max

Example: Find the box (without a top) with least surface area for a fixed volume.

Another way to solve this problem is by using implicit differentiation. As before, this method has some advantages and some disadvantages.

We start the same way:

$$V = x^2 y, \quad A = x^2 + 4xy$$

The goal is to find the minimum value of A while holding V constant.

Next, we just differentiate:

$$\frac{d}{dx} V = 2xy + x^2 \frac{dy}{dx} \Longrightarrow 0 = 2xy + x^2 y'$$

So $y' = -\frac{2y}{x}$.

$$\frac{dA}{dx} = 2x + 4y + 4xy'$$

And when we plug in $y' = -\frac{2y}{x}$ we get:

$$\begin{aligned} \frac{dA}{dx} &= 2x + 4y + 4x\left(-\frac{2y}{x}\right) \\ &= 2x + 4y - 8y \\ \frac{dA}{dx} &= 2x - 4y \end{aligned}$$

To find the critical points, we set $\frac{dA}{dx}$ equal to zero and get $0 = 2x - 4y$ or

$$\frac{x}{y} = 2.$$

This method gets to the answer faster and gets the nicer answer — the scale invariant proportions.

The disadvantage is that we did not check whether this critical point is a maximum, minimum, or neither.

Question: How would we check it?

Answer: By looking at the values of $A(0^+)$ and $A(\infty)$ or perhaps by using your intuition — would a very tall box with a tiny base have more or less surface area than a box that's the lower half of a cube? What about a very short box with a wide base?