Summing the Geometric Series

In lecture we saw a geometric argument that $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$. By answering the questions below, we complete an algebraic proof that this is true.

We start by proving by induction that:

$$S_N = \sum_{n=0}^{N} \frac{1}{2^n} = \frac{2^{N+1} - 1}{2^N}.$$

Finally we show that $\lim_{N \to \infty} S_N = 2$.

a) (Base case) Prove that $S_0 = \frac{2^1 - 1}{2^0} = 1$.

b) (Inductive hypothesis and inductive step) Assume that:

$$S_{N-1} = \frac{2^{(N-1)+1} - 1}{2^{N-1}} = \frac{2^N - 1}{2^{N-1}}.$$

Add $\frac{1}{2^N}$ to both sides to prove that:

$$S_N = \frac{2^{N+1} - 1}{2^N}.$$

This completes the inductive proof.

c) Show that if $S_N = \frac{2^{N+1} - 1}{2^N}$, then $\lim_{N \to \infty} S_N = 2$.

Solution

Students often find proof by induction intimidating or confusing. However, once one masters the technique, one finds that the hardest part of a proof can be the high school level algebraic manipulation.

a) (Base case) Prove that $S_0 = \frac{2^1 - 1}{2^0} = 1$.

The hardest part of this step is understanding and using the summation notation.

$$S_0 = \sum_{n=0}^{0} \frac{1}{2^n} = \frac{1}{2^0} = 1.$$

b) (Inductive hypothesis and inductive step) Assume that:

$$S_{N-1} = \frac{2^{(N-1)+1} - 1}{2^{N-1}} = \frac{2^N - 1}{2^{N-1}}.$$

Add $\frac{1}{2^N}$ to both sides to prove that:

$$S_N = \frac{2^{N+1}-1}{2^N}.$$

This completes the inductive proof.

This step requires careful manipulation of rational expressions, but is otherwise straightforward. We start with:

$$S_{N-1} = \frac{2^N - 1}{2^{N-1}}.$$

Applying the definition of S_N to the left hand side, we get:

$$\sum_{n=0}^{N-1} \frac{1}{2^n} = \frac{2^N - 1}{2^{N-1}}$$

Next we add $\frac{1}{2^N}$ to both sides:

$$\sum_{n=0}^{N-1} \frac{1}{2^n} + \frac{1}{2^N} = \frac{2^N - 1}{2^{N-1}} + \frac{1}{2^N}$$

Expand the summation and then add to complete the proof.

$$\begin{aligned}
\sum_{n=0}^{N-1} \frac{1}{2^n} + \frac{1}{2^N} &= \frac{2^N - 1}{2^{N-1}} + \frac{1}{2^N} \\
\frac{1}{2^0} + \frac{1}{2^2} + \cdots + \frac{1}{2^{N-1}} + \frac{1}{2^N} &= \frac{2(2^N - 1)}{2 \cdot 2^{N-1}} + \frac{1}{2^N} \\
S_N &= \frac{2^{N+1} - 2}{2^N} + \frac{1}{2^N} \qquad \text{(dfn. of } S_N\text{)} \\
S_N &= \frac{2^{N+1} - 1}{2^N}
\end{aligned}$$

c) Show that if $S_N = \frac{2^{N+1}-1}{2^N}$, then $\lim_{N\to\infty} S_N = 2$.

This is a straightforward evaluation of a limit.

$$\begin{aligned}
\lim_{N\to\infty} S_N &= \lim_{N\to\infty} \frac{2^{N+1}-1}{2^N} \\
&= \lim_{N\to\infty} \left(\frac{2^{N+1}}{2^N} - \frac{1}{2^N} \right) \\
&= \lim_{N\to\infty} \left(2 - \frac{1}{2^N} \right) \\
&= 2 - \lim_{N\to\infty} \frac{1}{2^N} \\
&= 2
\end{aligned}$$

To summarize, we first proved that $\frac{1}{2^0} + \frac{1}{2^2} + \cdots + \frac{1}{2^{N-1}} + \frac{1}{2^N} = \frac{2^{N+1}-1}{2^N}$. Then we showed that as the number of terms in this sum approaches infinity, the value of the sum approaches 2.