Remarks on Notation

We've been working with notation like ds^2 for a while now; what does this mean, what operations can we legitimately perform with these infinitesimals, and what isn't valid?

The basis for our arc length formula is that:

$$\Delta s^2 \approx \Delta x^2 + \Delta y^2.$$

We'll now see how our formula:

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

for parametric arc length can be more rigorously derived from the same basis.

Because Δt is not quite equal to 0, we can start by dividing both sides of the formula by Δt^2:

$$\begin{aligned} \Delta s^2 &\approx \Delta x^2 + \Delta y^2 \\ \left(\frac{\Delta s}{\Delta t}\right)^2 &\approx \left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 \end{aligned}$$

Finally, we take the limit as t goes to zero of both sides to conclude that:

$$\left(\frac{ds}{dx}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2.$$

(This is what derivatives are all about.)

Warning: Never write $\left(\frac{dx}{dt}\right)^2 = (x'(t))^2$ as $\frac{dx^2}{dt^2}$. If you do, it could be incorrectly interpreted to mean $\frac{d^2x}{dt^2} = x''(t)$.

Another unfortunate thing is that we write $\sin^2 x$ to mean $(\sin x)^2$, perhaps because typographers are lazy. There is inconsistency in mathematical notation, and we have to work with the conventions that exist.