Integration by “Advanced Guessing”

Example: $\displaystyle\int \frac{xdx}{\sqrt{1+x^2}}$

If we use the method of substitution, we start by setting u equal to the ugliest part of our integral:

$$u = 1 + x^2 \quad \text{and} \quad du = 2xdx.$$

The calculation looks like:

$$\begin{aligned}
\int \frac{xdx}{\sqrt{1+x^2}} &= \int \frac{\frac{1}{2}du}{\sqrt{u}} \\
&= \int \frac{u^{-\frac{1}{2}}}{2} du \\
&= 2\frac{u^{\frac{1}{2}}}{2} + c \\
&= u^{\frac{1}{2}} + c \\
&= (1+x^2)^{\frac{1}{2}} + c \\
&= \sqrt{1+x^2} + c
\end{aligned}$$

A better way to compute this is what we call “advanced guessing”. Once you’ve done enough of these problems that you know what’s going to happen, you can look at the $\sqrt{1+x^2}$ in the denominator and guess that the answer will involve $(1+x^2)^{1/2}$. Once you’ve made a guess, differentiate it and see if it works!

$$\begin{aligned}
\frac{d}{dx}(1+x^2)^{\frac{1}{2}} &= \frac{1}{2}(1+x^2)^{-\frac{1}{2}}(2x) \\
&= (1+x^2)^{-\frac{1}{2}}(x) \\
&= \frac{x}{\sqrt{1+x^2}}
\end{aligned}$$

As you can see, using this method we quickly confirm that:

$$\int \frac{xdx}{\sqrt{1+x^2}} = (1+x^2)^{1/2} + c.$$

This method is highly recommended, but it takes some getting used to.

Example: $\displaystyle\int e^{6x}dx$

We know that the derivative of e^x is e^x, so we guess e^{6x}. Then we check our guess using the chain rule:

$$\frac{d}{dx}(e)^{6x} = e^{6x}(6) = 6e^{6x}$$

This has a multiple of 6 that's not in the integral we're trying to compute, so we should divide our guess by 6 to get the correct answer:

$$\int e^{6x} dx = \frac{1}{6} e^{6x} + c.$$

We could also have used the substitution $u = 6x$. It would have worked, but it would have taken much longer.