Can Design

There are many factors to consider in food packaging, including marketing, durability, cost and materials. In this problem we minimize the cost of materials for a can.

Find the height and radius that minimizes the surface area of a can whose volume is 1 liter = 1000 cm^3.

Solution

We start by drawing a sketch of the can.

The formula for its volume is (area of base) · (height) = $\pi r^2 h$. Its surface consists of three parts — top, bottom and sides. The surface areas of the top and bottom are πr^2. The surface area of the sides is found by multiplying the circumference of the bottom by the height of the can; it is $2\pi r h$.

We now have the two equations:

$$\begin{aligned} V = 1 \text{ liter} &= \pi r^2 h \\ SA &= 2\pi r^2 + 2\pi r h. \end{aligned}$$

We wish to minimize the surface area, which suggests taking a derivative. Since h is dependent on r we cannot directly take the derivative of SA directly yet. We solve the volume equation for h:

$$\begin{aligned} 1 &= \pi r^2 h \\ h &= \frac{1}{\pi r^2}. \end{aligned}$$

We can now plug this expression in to the surface area equation and take the derivative with respect to r.

$$\begin{aligned} SA &= 2\pi r^2 + 2\pi r h \\ &= 2\pi r^2 + 2\pi r \frac{1}{\pi r^2} \\ SA &= 2\pi r^2 + \frac{2}{r} \\ \frac{d}{dr} SA &= 4\pi r - \frac{2}{r^2} \end{aligned}$$

To find the critical points, we set the derivative equal to zero and solve for r:

$$\begin{aligned} \frac{d}{dr} SA &= 0 \\ 4\pi r - \frac{2}{r^2} &= 0 \\ 4\pi r &= \frac{2}{r^2} \end{aligned}$$

$$\begin{aligned} r^3 &= \frac{1}{2\pi} \\ r &= \frac{1}{\sqrt[3]{2\pi}}. \end{aligned}$$

We now know that there is a critical point at $r = \frac{1}{\sqrt[3]{2\pi}}$, but we do not know if this corresponds to a minimum surface area. We must also consider the "end points" of the graph of our function.

As r approaches 0, the can is increasingly tall and thin; the area of the sides of the can approaches infinity. This is not the best solution.

As r increases the surface area $2\pi r^2 + \frac{2}{r}$ goes to infinity, so the minimum is not at this endpoint either.

We conclude that the critical point $r = \frac{1}{\sqrt[3]{2\pi}}$ probably represents the minimum surface area. To be sure, we calculate h and estimate the size of the can.

$$\begin{aligned} h &= \frac{1}{\pi r^2} \\ &= \frac{1}{\pi(\frac{1}{\sqrt[3]{2\pi}})^2} \\ h &= \sqrt[3]{\frac{4}{\pi}} \end{aligned}$$

One liter is 1000 cubic centimeters, so $r = \sqrt[3]{\frac{1}{2\pi} \text{ liters}} \approx 5$ cm and $h \approx 11$ cm. This is about the same size as the bottom half of a 2 liter bottle, so seems like a reasonable answer.