Probability Function

A Poisson process is a situation in which a phenomenon occurs at a constant average rate. Each occurrence is independent of all other occurrences; in a Poisson process, an event does not become more likely to occur just because it's been a long time since its last occurrence. The location of potholes on a highway or the emission over time of particles from a radioactive substance may be Poisson processes.

The probability density function:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0 \end{cases}$$

describes the relative likelihood of an occurrence at time or position x, where λ describes the average rate of occurrence.

The probability $P(a < x < b)$ of an event occurring in the interval between a and b is given by:

$$\int_a^b f(x)\,dx.$$

Compute this integral:

a) for the case in which a and b are both positive (assume $a < b$),

b) for the case in which $a \leq 0$ and $b > 0$,

c) for the case in which $a \leq 0$ and $b \leq 0$.

Solution

The solutions appear in reverse order of complexity; you may wish to start with the last and work back to the first.

a) Calculate $\int_a^b f(x)\,dx$ for the case in which a and b are both positive (assume $a < b$).

Because $x > 0$ over the entire interval from a to b,

$$\int_a^b f(x)\,dx = \int_a^b \lambda e^{-\lambda x}\,dx.$$

The antiderivative of $\lambda e^{-\lambda x}$ is $-e^{-\lambda x}$; from this point the calculation is straightforward.

$$\begin{aligned} \int_a^b \lambda e^{-\lambda x}\,dx &= -e^{-\lambda x}\Big|_a^b \\ &= -e^{-\lambda b} - (-e^{-\lambda a}) \\ &= e^{-\lambda a} - e^{-\lambda b}. \end{aligned}$$

Given a Poisson distribution with rate parameter λ and $b \geq a > 0$, the probability of an event occurring in the interval $[a, b]$ is $e^{-\lambda a} - e^{-\lambda b}$.

b) Calculate $\int_a^b f(x)\, dx$ for the case in which $a \leq 0$ and $b > 0$.

We haven't done many examples of integration of piecewise defined functions, but we can use the geometric interpretation of the definite integral to determine what we must do.

On the interval $[a, 0]$, $f(x) = 0$ so the area under that part of the graph of $f(x)$ is 0. On the interval $[b, 0]$, $f(x) = \lambda e^{-\lambda x}$. The area under the graph of $f(x)$ between a and b must be:

$$\begin{aligned} \int_a^b f(x)\, dx &= \int_a^0 f(x)\, dx + \int_0^b f(x)\, dx \\ &= 0 + \int_0^b \lambda e^{-\lambda x}\, dx \\ &= -e^{-\lambda x}\Big|_0^b \\ &= -e^{-\lambda b} - (-e^{-\lambda \cdot 0}) \\ &= 1 - e^{-\lambda b}. \end{aligned}$$

c) Calculate $\int_a^b f(x)\, dx$ for the case in which $a \leq 0$ and $b \leq 0$.

This calculation is trivial. On the interval $[a, b]$, $f(x) = 0$. The area between the graph of f and the x-axis is 0, so:

$$\int_a^b f(x)\, dx = 0 \qquad \text{when } a, b \leq 0.$$