Detailed Example of Curve Sketching

Example Sketch the graph of $f(x) = \dfrac{x}{\ln x}$. (Note: this function is only defined for $x > 0$)

1. Plot

   a The function is discontinuous at $x = 1$, because $\ln 1 = 0$.

   $$f(1^+) = \frac{1}{\ln 1^+} = \frac{1}{0^+} = \infty$$

   $$f(1^-) = \frac{1}{\ln 1^-} = \frac{1}{0^-} = -\infty$$

   b endpoints (or $x \to \pm\infty$)

   $$f(0^+) = \frac{0^+}{\ln 0^+} = \frac{0^+}{-\infty} = 0$$

   The situation is a little more complicated at the other end; we'll get a feel for what happens by plugging in $x = 10^{10}$.

   $$f(10^{10}) = \frac{10^{10}}{\ln 10^{10}} = \frac{10^{10}}{10 \ln 10} = \frac{10^9}{\ln 10} >> 1$$

   We conclude that $f(\infty) = \infty$.

We can now start sketching our graph. The point $(0, 0)$ is one endpoint of the graph. There's a vertical asymptote at $x = 1$, and the graph is descending before and after the asymptote. Finally, we know that $f(x)$ increases to positive infinity as x does. We already have a pretty good idea of what to expect from this graph!

2. Find the critical points

   $$\begin{aligned} f'(x) &= \frac{1 \cdot \ln x - x\left(\frac{1}{x}\right)}{(\ln x)^2} \\ &= \frac{(\ln x) - 1}{(\ln x)^2} \end{aligned}$$

   a $f'(x) = 0$ when $\ln x = 1$, so when $x = e$. This is our only critical point.

   b $f(e) = \frac{e}{\ln e} = e$ is our critical value. The point (e, e) is a critical point on our graph; we can label it with the letter c. (It's ok if our graph is not to scale; we'll do the best we can.)

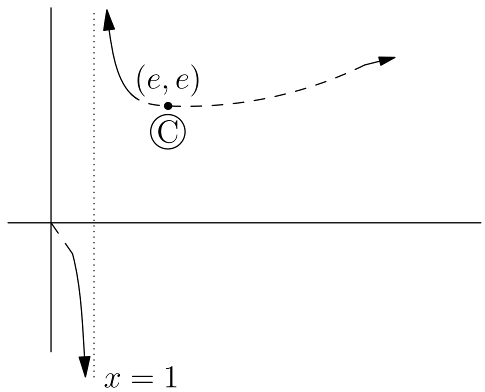

Figure 1: Sketch using starting point, asymptote, critical point and endpoints.

We now know the qualitative behavior of the graph. We know exactly where f is increasing and decreasing because the graph can only change direction at critical points and discontinuities; we've identified all of those. The rest is more or less decoration.

3. Double check using the sign of f'.

   We already know:

$$\begin{array}{ccc} f & \text{is decreasing on} & 0 < x < 1 \\ f & \text{is decreasing on} & 1 < x < e \\ f & \text{is increasing on} & e < x < \infty \end{array}$$

   We now double check this.

$$f'(x) = \frac{(\ln x) - 1}{(\ln x)^2}$$

   When x is between 0 and 1, $f'(x)$ equals a negative number divided by a positive number so is negative.

   When x is between 1 and e, $f'(x)$ again equals a negative number divided by a positive number so is negative.

   When x is between e and ∞, $f'(x)$ equals a positive number divided by a positive number so is positive.

   This confirms what we learned in steps 1 and 2.

   Sometimes steps 1 and 2 will be harder; then you might need to do this step first to get a feel for what the graph looks like.

   There's one more piece of information we can get from the first derivative $f'(x) = \frac{(\ln x)-1}{(\ln x)^2}$. It's possible for the denominator to be infinite; this is another situation in which the derivative is zero. So $f'(0^+) = 0$ and $x = 0^+$ is another critical point with critical value $-\infty$.

An easier way to see this is to rewrite $f'(x)$ as:

$$\frac{1}{\ln x} - \frac{1}{(\ln x)^2}$$

and note that:

$$f'(0^+) = \frac{1}{\ln 0^+} - \frac{1}{(\ln 0^+)^2} = \frac{1}{-\infty} - \frac{1}{(\infty)^2} = 0 - 0 = 0.$$

4. Use $f''(x)$ to find out whether the graph is concave up or concave down.

$$f'(x) = (\ln x)^{-1} - (\ln x)^{-2}$$

So

$$\begin{aligned} f''(x) &= -(\ln x)^{-2}\frac{1}{x} - -2(\ln x)^{-3}\frac{1}{x} \\ &= \frac{-(\ln x)^{-2} + 2(\ln x)^{-3}}{x}\frac{(\ln x)^3}{(\ln x)^3} \\ &= \frac{-\ln x + 2}{x(\ln x)^3} \\ f''(x) &= \frac{2 - \ln x}{x(\ln x)^3} \end{aligned}$$

We need to figure out where this is positive or negative. There are two places where the sign might change – when $2 - \ln x$ changes sign or when $(\ln x)^3$ changes sign. (Remember x will always be positive.)

The value of $2 - \ln x$ is positive when $\ln x < 2$ (when $x < e^2$) and negative when $x > e^2$. The denominator is positive when $x > 1$ and negative when $x < 1$. Combining these, we get:

$$\begin{aligned} 0 < x < 1 &\implies f''(x) < 0 \text{ (concave down)} \\ 1 < x < e^2 &\implies f''(x) > 0 \text{ (concave up)} \\ e^2 < x < \infty &\implies f''(x) < 0 \text{ (concave down)} \end{aligned}$$

This means that there's a "wiggle" at the point $\left(e^2, \frac{e^2}{2}\right)$ on the graph. The value of $f(x)$ is still increasing and the graph continues to rise, but the graph is rising less and less steeply as the values of $f'(x)$ decrease.

5. Combine this information to draw the graph.

   We've been doing this as we go. If you're working a homework problem, at this point you might copy your graph to a clean sheet of paper.

This is probably as detailed a graph as we'll ever draw. In fact, one advantage of our next topic is that it will reduce the need to be this detailed.

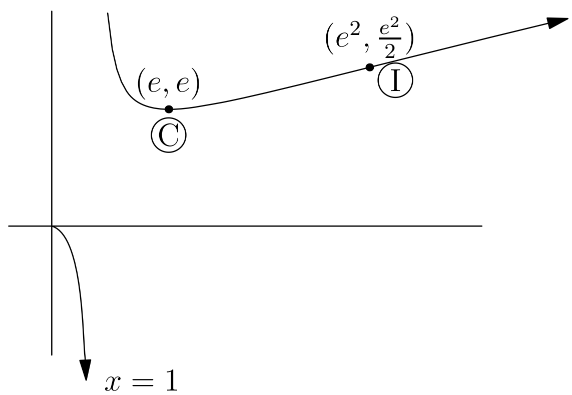

Figure 2: Final sketch.