

Revolution About the x -axis

Find the volume of the solid of revolution generated by rotating the regions bounded by the curves given around the x -axis.

a) $y = 3x - x^2, y = 0$

b) $y = \sqrt{ax}, y = 0, x = a$

Solutions

a) $y = 3x - x^2, y = 0$

We start by sketching the region, which is a parabolic “hump” between $x = 0$ and $x = 3$ with vertex $(1.5, 2.25)$.

We could compute the volume of the solid by using shells or disks. Because we’re given the equation $y = 3x - x^2$ for y in terms of x , it seems easiest to use disks. The radius of each disk will equal the height of the curve, $3x - x^2$ so a disk’s volume will be $\pi(3x - x^2)^2 dx$. The limits of integration are the x -intercepts (crossing points) $x = 0$ and $x = 3$.

The volume of the solid of revolution is:

$$\begin{aligned} \int_0^3 \pi(3x - x^2)^2 dx &= \pi \int_0^3 9x^2 - 6x^3 + x^4 dx \\ &= \pi \left[3x^3 - \frac{3}{2}x^4 + \frac{1}{5}x^5 \right]_0^3 \\ &= \pi \left[\left(3 \cdot 3^3 - \frac{3}{2} \cdot 3^4 + \frac{1}{5} \cdot 3^5 \right) - 0 \right] \\ &= 3^4 \pi \left(1 - \frac{3}{2} + \frac{3}{5} \right) \\ &= \frac{3^4 \pi}{10} \approx 8\pi. \end{aligned}$$

This seems like a reasonable answer; 8π is the volume of a cylinder with radius 2 and height 2.

b) $y = \sqrt{ax}, y = 0, x = a$

We must either sketch this region by hand or choose a value for a when using an electronic graphing tool. Either way, we find that the region has the shape of half of a parabolic “hump”, opening to the right, with vertex at $(0, 0)$. The right hand boundary of the region is the line segment joining $(a, 0)$ and (a, a) .

Again we may use either washers or shells to compute the volume of the solid of revolution. The calculation is probably slightly simpler using washers, but we’ll use shells in this solution and integrate with respect to y for variety.

The extent of each shell will be from the curve $y = \sqrt{ax}$ (equivalently $x = \frac{y^2}{a}$) to the line $x = a$. The volume of each shell is $2\pi y \left(a - \frac{y^2}{a}\right) dy$. The region extends from $y = 0$ to $y = a$, so the volume of revolution is:

$$\begin{aligned} \int_0^a 2\pi y \left(a - \frac{y^2}{a}\right) dy &= 2\pi \int_0^a \left(ay - \frac{y^3}{a}\right) dy \\ &= 2\pi \left[\frac{ay^2}{2} - \frac{y^4}{4a}\right]_0^a \\ &= \pi \left[\left(a^3 - \frac{a^3}{2}\right) - 0\right] \\ &= \frac{\pi a^3}{2}. \end{aligned}$$

If we remember that the volume of a cone with radius a and height a is $\frac{\pi a^3}{3}$, we can feel relatively confident that our answer is correct. Those who don't know the formula for the volume of a cone might compare their answer to the volume of a suitable cylinder.

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