

Max/Min Example 2

This is an example of a minimization problem with a constraint.

Example: Find the box (without a top) with least surface area for a fixed volume.

Again, we start by drawing a diagram and choosing variables. (This time we'll have four variable names.) I'll make things simpler by telling you in advance that the best box has a square bottom — knowing that the length of the box equals its width will let us use one fewer variable. It's often true that the correct answer is something symmetric, but you weren't expected to know this in advance.

Draw a picture of an open-topped box whose length and width equals x and whose height is y . The volume of the box is $V = x^2y$ and its surface area is the area of the base plus the area of each of the four sides: $A = x^2 + 4(xy)$.

The big difference between this problem and the last is that we have the *constraint* that the box must have a certain volume — this determines the relationship between x and y .

$$y = \frac{V}{x^2}$$

We can use this to rewrite the formula for A in terms of a single variable:

$$\begin{aligned} A(x) &= x^2 + 4x \frac{V}{x^2} \\ A(x) &= x^2 + \frac{4V}{x} \end{aligned}$$

Now that we have a formula for the value we're trying to minimize — the surface area — we'll follow the same procedure as before. Namely, we'll look for the critical points then check the endpoints and any discontinuities.

To find the critical points we take the derivative of $A(x)$ and set it equal to zero.

$$\begin{aligned} A(x) &= x^2 + \frac{4V}{x} \\ A'(x) &= 2x - \frac{4V}{x^2} \\ 2x - \frac{4V}{x^2} &= 0 \\ 2x &= \frac{4V}{x^2} \\ x &= \frac{2V}{x^2} \\ x^3 &= 2V \\ x &= 2^{\frac{1}{3}}V^{\frac{1}{3}} \quad (\text{critical point}) \end{aligned}$$

We are not done; we don't even know whether this critical point gives us the most surface area or the least.

Let's check the ends: what are they? What's the smallest possible value of x ?

Student: $x > 0$

This is a good answer! We can't build a box with a negative side length.

What's the largest possible value of x ? It's true that:

$$x < \sqrt{\frac{V}{y}}$$

but since we said $y = \frac{V}{x^2}$ we can't use this as a limit.

Student: x is less than infinity?

That's right. $0 < x < \infty$.

This is an important realization; if there's no obvious limit on a variable the upper limit is infinity. And infinity is a very important endpoint, and is usually an easy endpoint to check.

We have to consider the possibility that if we shrink the sides all the way to $x = 0$ we'll get a better box. It would be very strange — maybe infinitely tall — but it might have the least surface area; we'll have to see.

$$A(0^+) = \left(x^2 + \frac{4V}{x} \right) \Big|_{x=0^+}$$

Here the term $\frac{4V}{x}$ is going to infinity as x approaches 0 from the positive side; a box whose surface area approaches infinity is a bad box.

At the other end we get:

$$A(\infty) = \left(x^2 + \frac{4V}{x} \right) \Big|_{x=\infty}$$

which is also infinite.

We can now draw a schematic for our graph. The surface area goes to infinity as x goes to 0 (from the positive side) and as x goes to infinity. There's only one critical point — only one place where the graph can turn around — so that point must be where the surface area reaches its minimum.

An alternative to checking ends is the *second derivative test*. The second derivative test is not recommended, and for most of the problems in this class it will be hard to use the second derivative test. However, this is a simple enough problem that we are able to use it.

$$\begin{aligned} A'(x) &= 2x - \frac{4V}{x^2} \\ A''(x) &= 2 + \frac{8V}{x^3} \end{aligned}$$

The second derivative is always positive (because x is always positive), so the graph of $A(x)$ is always concave up. This tells us that the critical point is at the lowest point of a “smile shape” and so must correspond to a minimum value of $A(x)$.

Question: Is this the answer to the question, or do we have to calculate y and A and so on?

Answer: What answer is appropriate depends on what the question is. We know that $x = 2^{\frac{1}{3}}V^{\frac{1}{3}}$. If we’re going to build the box we’ll also need to know the y value, which is the height of the box.

$$\begin{aligned} y &= \frac{V}{x^2} \\ &= \frac{V}{(2^{\frac{1}{3}}V^{\frac{1}{3}})^2} \\ y &= 2^{-\frac{2}{3}}V^{\frac{1}{3}} \end{aligned}$$

We could figure out the value of A ; this would be appropriate if we wanted to know how much money it was going to cost to build this box.

$$\begin{aligned} A(x) &= x^2 + \frac{4V}{x} \\ &= (2^{\frac{1}{3}}V^{\frac{1}{3}})^2 + \frac{4V}{2^{\frac{1}{3}}V^{\frac{1}{3}}} \\ &= 2^{\frac{2}{3}}V^{\frac{2}{3}} + 4V \cdot 2^{-\frac{1}{3}}V^{-\frac{1}{3}} \\ &= 2^{\frac{2}{3}}V^{\frac{2}{3}} + (2 \cdot 2)2^{-\frac{1}{3}}V^{\frac{2}{3}} \\ &= (2^{\frac{2}{3}} + 2 \cdot 2^{\frac{2}{3}})V^{\frac{2}{3}} \\ &= 3 \cdot 2^{\frac{2}{3}}V^{\frac{2}{3}} \end{aligned}$$

So one possible answer to this problem is that the box with minimum surface area has:

$$\begin{aligned} \text{length} &= 2^{\frac{1}{3}}V^{\frac{1}{3}} \\ \text{height} &= 2^{-\frac{2}{3}}V^{\frac{1}{3}} \text{ and} \\ \text{surface area} &= 3 \cdot 2^{\frac{2}{3}}V^{\frac{2}{3}} \end{aligned}$$

However, there are much more meaningful ways of answering this question. If we use *dimensionless variables* it won’t make a difference whether the sides of the box are measured in inches or kilometers and we may learn more about the problem. One famous dimensionless quantity is π , the ratio of the circumference of a circle to its diameter.

For example, $\frac{A}{V^{\frac{2}{3}}} = 3 \cdot 2^{\frac{1}{3}}$ is a dimensionless quantity, because if A is measured in m^2 and V is measured in m^3 the dimensions will cancel. If you increase the volume, you'll increase the surface area by the $\frac{2}{3}$ power of the volume.

The other dimensionless quantity is:

$$\frac{x}{y} = \frac{2^{\frac{1}{3}} V^{\frac{1}{3}}}{2^{-\frac{2}{3}} V^{\frac{1}{3}}} = 2$$

This is the best answer to the question; it tells us that the box with minimum surface area has a base that is twice as wide as its height. This is the optimal shape of the box.

Question: Could we have gotten the answer if we weren't told that the bottom was square?

Answer: Yes. You can do it all in one step if you use multivariable calculus and include variables for both the length and width of the box. Or you can do it in two steps using ideas from this class by first figuring out what rectangle has the smallest perimeter for a fixed area.

Question: Why did you divide x by y ?

Answer: We were looking for dimensionless quantities. The lengths x and y are measured in the same units, and x/y gives you the proportions of the box. Another word for what we're interested in here is *proportions*. These are universal, independent of the volume V . The proportions would be the same for any box, at any scale; this is why they provide the nicest answer.

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