Improper Integrals of the Second Kind, Continued

We'll continue our discussion of integrals of functions which have singularities at finite values; for example, $f(x) = \frac{1}{x}$. If $f(x)$ has a singularity at 0 we define

$$\int_0^1 f(x)\,dx = \lim_{a\to 0^+} \int_a^1 f(x)\,dx.$$

As before, we say the integral *converges* if this limit exists and *diverges* if not.

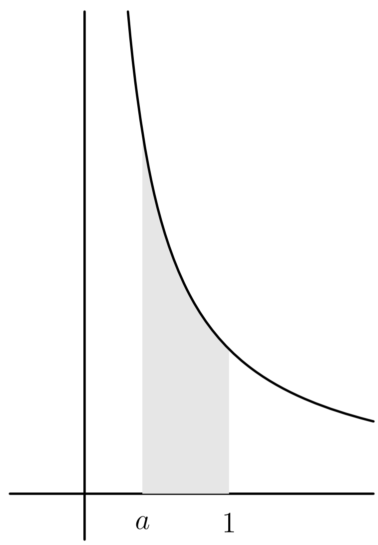

Figure 1: Area under the graph of $y = \frac{1}{x}$.

We treat this infinite vertical "tail" the same way we treated horizontal tails. Figure 1 shows a function whose value goes to positive infinity as x goes to zero from the right hand side. We don't know whether the area under its graph between 0 and 1 is going to be infinite or finite, so we cut it off at some point a where we know it will be finite. Then we let a go to zero from above ($a \to 0^+$) and see whether the area under the curve between a and 1 goes to infinity or to some finite limit.

Example: $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$

$$\begin{aligned}
\int_0^1 \frac{dx}{\sqrt{x}} &= \int_0^1 x^{-1/2}\,dx \\
&= \frac{1}{1/2} x^{1/2}\Big|_0^1 \\
&= 2x^{1/2}\Big|_0^1
\end{aligned}$$

$$
\begin{aligned}
&= 2 \cdot 1^{1/2} - 2 \cdot 0^{1/2} \\
&= 2.
\end{aligned}
$$

This is a convergent integral.

Example: $\displaystyle\int_0^1 \frac{dx}{x}$

$$
\begin{aligned}
\int_0^1 \frac{dx}{x} &= \ln x\big|_0^1 \\
&= \ln 1 - \ln 0^+ \\
&= 0 - (-\infty) \\
&= +\infty.
\end{aligned}
$$

This integral diverges.

In general:

$$
\begin{aligned}
\int_0^1 \frac{dx}{x^p} &= \left.\frac{x^{-p+1}}{-p+1}\right|_0^1 \quad \text{(for } p \neq 1\text{)} \\
&= \frac{1^{-p+1}}{-p+1} - \frac{0^{-p+1}}{-p+1} \\
&= \begin{cases} \dfrac{1}{1-p} & \text{if } p < 1 \\ \text{diverges} & \text{if } p \geq 1. \end{cases}
\end{aligned}
$$