The Mean Value Theorem and Inequalities

The mean value theorem tells us that if f and f' are continuous on $[a, b]$ then:

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

for some value c between a and b. Since f' is continuous, $f'(c)$ must lie between the minimum and maximum values of $f'(x)$ on $[a, b]$. In other words:

$$\min_{a \le x \le b} f'(x) \le \frac{f(b) - f(a)}{b - a} = f'(c) \le \max_{a \le x \le b} f'(x).$$

This is the form that the mean value theorem takes when it is used in problem solving (as opposed to mathematical proofs), and this is the form that you will need to know for the test.

In practice, you may even forget the mean value theorem and remember only these three inequalities:

- If $f'(c) > 0$ then $f(b) > f(a)$.
- If $f'(c) < 0$ then $f(b) < f(a)$.
- If $f'(c) = 0$ then $f(b) = f(a)$.

These can be used to prove mathematical inequalities. The following examples compare the function e^x to its linear and quadratic approximations and are the first steps toward a deeper understanding of the function.

Example: Show that $e^x > 1 + x$ for $x > 0$.

To prove this, we'll instead show that $f(x) = e^x - (1 + x)$ is always positive. We know that $f(0) = e^0 - (1 + 0) = 0$ and $f'(x) = e^x - 1$. When x is positive, $f'(x)$ is positive because $e^x > 1$.

We know that if $f'(x) > 0$ on an interval then $f(x)$ is increasing on that interval, so we can conclude that $f(x) > f(0)$ for $x > 0$. In other words,

$$e^x - (1 + x) > 0 \iff e^x > 1 + x.$$

Example: Show that $e^x > 1 + x + \frac{x^2}{2}$ for $x > 0$.

The value of $1 + x + \frac{x^2}{2}$ is slightly greater than that of $1 + x$, but it turns out that it's still less than the value of e^x. We let $g(x) = e^x - (1 + x + \frac{x^2}{2})$ and do the same thing we did before:

$$\begin{aligned} g(0) &= 1 - (1) = 0 \\ g'(x) &= e^x - (1 + x) \end{aligned}$$

We know $g'(x) > 0$ because we proved $f(x) > 0$ in the above example. Since $g'(x)$ is positive, g is increasing for $x > 0$, so $g(x) > g(0)$ when $x > 0$, so $e^x - (1 + x + \frac{x^2}{2}) > 0$ and $e^x > (1 + x + \frac{x^2}{2})$.

We can keep on going: $e^x > 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}$ for $x > 0$. Eventually, it turns out that

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \quad \text{(an infinite sum)}$$

We will be discussing this when we get to Taylor series near the end of the course.