Summing the Geometric Series

In lecture we saw a geometric argument that $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$. By answering the questions below, we complete an algebraic proof that this is true.

We start by proving by induction that:

$$S_N = \sum_{n=0}^{N} \frac{1}{2^n} = \frac{2^{N+1} - 1}{2^N}.$$

Finally we show that $\lim_{N\to\infty} S_N = 2$.

a) (Base case) Prove that $S_0 = \frac{2^1-1}{2^0} = 1$.

b) (Inductive hypothesis and inductive step) Assume that:

$$S_{N-1} = \frac{2^{(N-1)+1} - 1}{2^{N-1}} = \frac{2^N - 1}{2^{N-1}}.$$

Add $\frac{1}{2^N}$ to both sides to prove that:

$$S_N = \frac{2^{N+1} - 1}{2^N}.$$

This completes the inductive proof.

c) Show that if $S_N = \frac{2^{N+1} - 1}{2^N}$, then $\lim_{N\to\infty} S_N = 2$.