Comparison Tests

Integral Comparison

We used integral comparison when we applied Riemann sums to understanding $\sum_1^\infty \frac{1}{n}$ in terms of $\int_1^\infty \frac{dx}{x}$, and we've made several other comparisons between integrals and series in this lecture. Now we learn the general theory behind this technique.

Theorem: If $f(x)$ is decreasing and $f(x) > 0$ on the interval from 1 to infinity, then either the sum $\sum_1^\infty f(n)$ and the integral $\int_1^\infty f(x)\,dx$ both diverge or they both converge and:

$$\sum_{n=1}^{\infty} f(n) - \int_1^\infty f(x)\,dx \quad < \quad f(1).$$

For example, when $S_N = \sum_1^N \frac{1}{n}$ we showed that $|S_n - \ln N| < 1$.

Since it's very difficult to compute infinite sums and it's easy to compute indefinite integrals, this is an extremely useful theorem.

Limit Comparison

Theorem: If $f(n) \sim g(n)$ (i.e. if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$) and $g(n) > 0$ for all n, then either both $\sum_{n=1}^{\infty} f(n)$ and $\sum_{n=1}^{\infty} g(n)$ converge or both diverge.

This says that if f and g behave the same way in their tails, their convergence properties will be similar.