Related Rates, A Conical Tank

Example: Consider a conical tank whose radius at the top is 4 feet and whose depth is 10 feet. It's being filled with water at the rate of 2 cubic feet per minute. How fast is the water level rising when it is at depth 5 feet?

As always, our first step is to set up a diagram and variables.

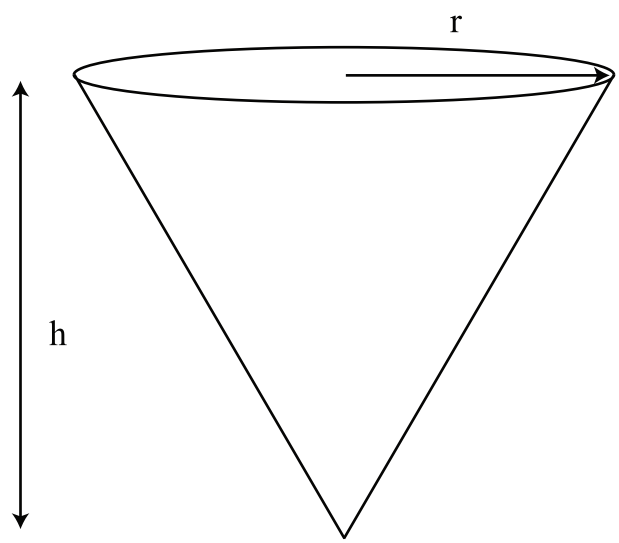

Figure 1: Illustration of example 2: inverted cone water tank.

This diagram just helps us to start thinking about the problem. For instance, we see that because the cone is narrower at the bottom the rate of change of the depth will vary; we need to depict the water level. We also realize that it's difficult to draw useful and accurate diagrams of three dimensional figures — a simple schematic may be more helpful.

The key here is to draw a two-dimensional cross-section. In the figure we're looking at one half of a vertical slice of the tank. The height of the slice equals 10 feet, which is the height of the tank. The widest part of the slice is 4 feet, which is the distance from center to edge of the top of the tank.

We'll use the variable r will represent the distance from center to edge of the top of the water, and h will represent the height of the top of the water (which is also the depth of the water). We can find the relationship between r and h from Fig. 2) using similar triangles:

$$\frac{r}{h} = \frac{4}{10}.$$

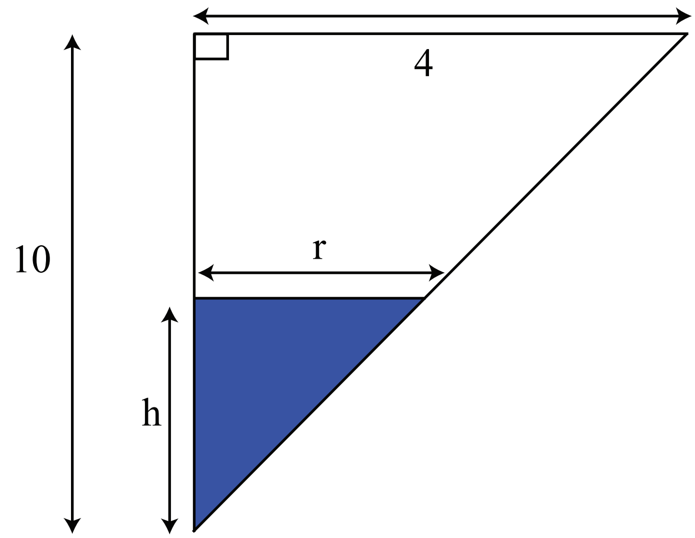

Figure 2: Relating r and h.

Our goal is to find out how fast the water is rising when the tank is half full. What we know is that the volume of water in the tank is changing at a rate of 2 cubic feet per minute. We need equations relating the volume of water in the tank to its depth, h.

The volume of a cone is $\frac{1}{3}\cdot\text{base}\cdot\text{height}$. From Fig. 1), the volume of this tank is given by:

$$V = \frac{1}{3}\cdot\underbrace{\pi r^2}_{\text{base}}\cdot\underbrace{h}_{\text{height}}$$

This relates the volume to the height and radius, and we know the relation between the hight and the radius. We have one more piece of information that we can use: $\frac{dV}{dt} = 2$.

The question is: "What is $\frac{dh}{dt}$ when $h = 5$?"

We've now translated all of the words in the original problem into formulas. Our word problem is now simply a calculus problem.

We could do this by implicit differentiation, but it's easy enough to solve for r in terms of h that there's no need to.

$$r = \frac{2}{5}h.$$

We plug this expression for r back into V to get:

$$V = \frac{1}{3}\pi\left(\frac{2}{5}h\right)^2 h = \frac{4}{3(25)}\pi h^3$$

At this point we could solve for h, but that turns out to be a bad idea. Implicit differentiation is much easier.

$$\begin{aligned}\frac{dV}{dt} &= \frac{dV}{dh}\frac{dh}{dt} \\ &= \frac{\pi}{3}\left(\frac{2}{5}\right)^2 3h^2\frac{dh}{dt} \\ &= \frac{4}{25}\pi h^2 h'\end{aligned}$$

Now that we've calculated the rates of change we can plug in the numbers $\frac{dV}{dt} = 2$ and $h = 5$:

$$2 = \left(\frac{4}{25}\right)\pi(5)^2 h'$$

$$2 = 4\pi h'$$

$$h' = \frac{1}{2\pi}\text{ft/min}$$

We were given the rate at which the volume of water in the tank was changing and we used that to compute the rate at which the water in the tank was rising. At the heart of this calculation was the chain rule:

$$\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt}.$$

Related rates problems are all about applying the chain rule to solve word problems.