Equation of an Off-Center Circle

This is a standard example that comes up a lot. Circles are easy to describe, unless the origin is on the rim of the circle. We'll calculate the equation in polar coordinates of a circle with center $(a, 0)$ and radius $(2a, 0)$. You should expect to repeat this calculation a few times in this class and then memorize it for multivariable calculus, where you'll need it often.

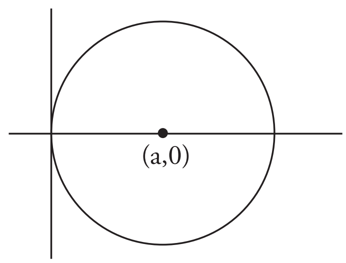

Figure 1: Off center circle through $(0, 0)$.

In rectangular coordinates, the equation of this circle is:

$$(x-a)^2 + y^2 = a^2.$$

We could plug in $x = r\cos\theta$, $y = \sin\theta$ to convert to polar coordinates, but there's a faster way. We start by expanding and simplifying:

$$\begin{aligned}
(x-a)^2 + y^2 &= a^2 \\
x^2 - 2ax + a^2 + y^2 &= a^2 \\
x^2 - 2ax + y^2 &= 0 \\
(x^2 + y^2) - 2ax &= 0 \\
r^2 - 2ar\cos\theta &= 0 \\
r^2 &= 2ar\cos\theta \\
\Longrightarrow r &= 2a\cos\theta \quad (\text{or } r = 0).
\end{aligned}$$

We used the facts that $x^2 + y^2 = r^2$ and $x = r\cos\theta$ to conclude that there were two values of r that satisfy this equation; $r = 2a\cos\theta$ and $r = 0$. These are the equations describing r in terms of θ that describe this circle in polar coordinates.

In order to use the equation $r = 2a\cos\theta$, we need to figure out the appropriate range of values for θ. By looking at the graph we see that $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Our final equation is:

$$r = 0 \qquad \text{or} \qquad r = 2a\cos\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

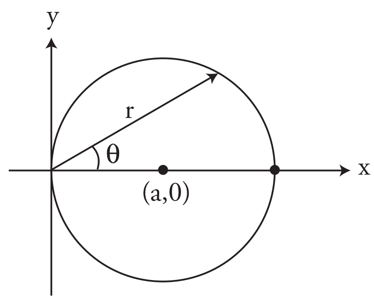

Figure 2: Off center circle in polar coordinates.

To check our work, let's find some points on this curve:

- At $\theta = 0$, $r = 2a$ and so $x = 2a$ and $y = 0$.
- At $\theta = \frac{\pi}{4}$, $r = 2a \cos \frac{\pi}{4} = a\sqrt{2}$. Hence $x = a$ and $y = a$.