18.02 Multivariable Calculus
Fall 2007

# 3. Double Integrals

## 3A. Double integrals in rectangular coordinates

**3A-1**

a) Inner: $6x^2y + y^2\Big]_{y=-1}^{1} = 12x^2$;  Outer: $4x^3\Big]_0^2 = 32$ .

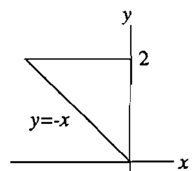
b) Inner: $-u\cos t + \frac{1}{2}t^2\cos u\Big]_{t=0}^{\pi} = 2u + \frac{1}{2}\pi^2\cos u$

Outer: $u^2 + \frac{1}{2}\pi^2 \sin u\Big]_0^{\pi/2} = (\frac{1}{2}\pi)^2 + \frac{1}{2}\pi^2 = \frac{3}{4}\pi^2$.

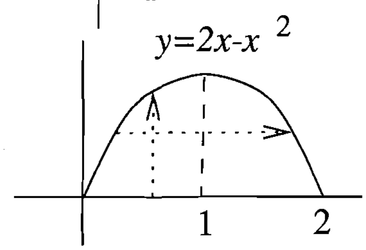
c) Inner: $x^2y^2\Big]_{\sqrt{x}}^{x^2} = x^6 - x^3$;  Outer: $\frac{1}{7}x^7 - \frac{1}{4}x^4\Big]_0^1 = \frac{1}{7} - \frac{1}{4} = -\frac{3}{28}$

d) Inner: $v\sqrt{u^2+4}\Big]_0^u = u\sqrt{u^2+4}$;  Outer: $\frac{1}{3}(u^2+4)^{3/2}\Big]_0^1 = \frac{1}{3}(5\sqrt{5} - 8)$

**3A-2**




a) (i) $\iint_R dy\,dx = \int_{-2}^{0}\int_{-x}^{2} dy\,dx$  (ii) $\iint_R dx\,dy = \int_0^2 \int_{-y}^{0} dx\,dy$

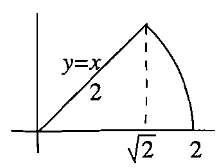



b) i) The ends of $R$ are at 0 and 2, since $2x - x^2 = 0$ has 0 and 2 as roots.

$$\iint_R dydx = \int_0^2 \int_0^{2x-x^2} dydx$$

ii) We solve $y = 2x - x^2$ for $x$ in terms of $y$: write the equation as $x^2 - 2x + y = 0$ and solve for $x$ by the quadratic formula, getting $x = 1 \pm \sqrt{1-y}$. Note also that the maximum point of the graph is $(1, 1)$ (it lies midway between the two roots 0 and 2). We get

$$\iint_R dxdy = \int_0^1 \int_{1-\sqrt{1-y}}^{1+\sqrt{1-y}} dxdy,$$




c) (i) $\iint_R dy\,dx = \int_0^{\sqrt{2}} \int_0^x dy\,dx + \int_{\sqrt{2}}^{2} \int_0^{\sqrt{4-x^2}} dy\,dx$

(ii) $\iint_R dx\,dy = \int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} dx\,dy$

d) Hint: First you have to find the points where the two curves intersect, by solving simultaneously $y^2 = x$ and $y = x - 2$ (eliminate $x$).

The integral $\iint_R dy\,dx$ requires two pieces; $\iint_R dx\,dy$ only one.

**3A-3** a) $\iint_R x\,dA = \int_0^2 \int_0^{1-x/2} x\,dy\,dx$;

Inner: $x(1 - \frac{1}{2}x)$  Outer: $\frac{1}{2}x^2 - \frac{1}{6}x^3\Big]_0^2 = \frac{4}{2} - \frac{8}{6} = \frac{2}{3}$.

b) $\displaystyle\iint_R (2x+y^2)\,dA = \int_0^1\int_0^{1-y^2}(2x+y^2)\,dx\,dy$

Inner: $x^2+y^2x\Big]_0^{1-y^2} = 1-y^2$; Outer: $y-\frac{1}{3}y^3\Big]_0^1 = \frac{2}{3}$.

c) $\displaystyle\iint_R y\,dA = \int_0^1\int_{y-1}^{1-y} y\,dx\,dy$

Inner: $xy\Big]_{y-1}^{1-y} = y[(1-y)-(y-1)] = 2y-2y^2$ Outer: $y^2-\frac{2}{3}y^3\Big]_0^1 = \frac{1}{3}$.

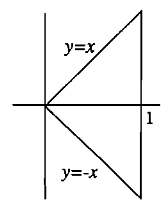
**3A-4** a) $\displaystyle\iint_R \sin^2 x\,dA = \int_{-\pi/2}^{\pi/2}\int_0^{\cos x}\sin^2 x\,dy\,dx$

Inner: $y\,\sin^2 x\Big]_0^{\cos x} = \cos x\sin^2 x$ Outer: $\frac{1}{3}\sin^3 x\Big]_{-\pi/2}^{\pi/2} = \frac{1}{3}(1-(-1)) = \frac{2}{3}$.

b) $\displaystyle\iint_R xy\,dA = \int_0^1\int_{x^2}^{x}(xy)\,dy\,dx.$

Inner: $\frac{1}{2}xy^2\Big]_{x^2}^{x} = \frac{1}{2}(x^3-x^5)$ Outer: $\displaystyle\frac{1}{2}\left(\frac{x^4}{4}-\frac{x^6}{6}\right)_0^1 = \frac{1}{2}\cdot\frac{1}{12} = \frac{1}{24}.$

c) The function $x^2-y^2$ is zero on the lines $y=x$ and $y=-x$, and positive on the region $R$ shown, lying between $x=0$ and $x=1$. Therefore



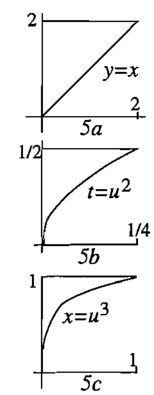

$$\text{Volume} = \iint_R (x^2-y^2)\,dA = \int_0^1\int_{-x}^{x}(x^2-y^2)\,dy\,dx.$$

Inner: $x^2y-\frac{1}{3}y^3\Big]_{-x}^{x} = \frac{4}{3}x^3$; Outer: $\frac{1}{3}x^4\Big]_0^1 = \frac{1}{3}$.

**3A-5** a) $\displaystyle\int_0^2\int_x^2 e^{-y^2}dy\,dx = \int_0^2\int_0^y e^{-y^2}dx\,dy = \int_0^2 e^{-y^2}\,y\,dy = -\frac{1}{2}e^{-y^2}\Big]_0^2 = \frac{1}{2}(1-e^{-4})$

b) $\displaystyle\int_0^{\frac14}\int_{\sqrt{t}}^{\frac12}\frac{e^u}{u}\,du\,dt = \int_0^{\frac12}\int_0^{u^2}\frac{e^u}{u}\,dt\,du = \int_0^{\frac12} u\,e^u\,du = (u-1)e^u\Big]_0^{\frac12} = 1-\frac{1}{2}\sqrt{e}$

c) $\displaystyle\int_0^1\int_{x^{1/3}}^1\frac{1}{1+u^4}\,du\,dx = \int_0^1\int_0^{u^3}\frac{1}{1+u^4}\,dx\,du = \int_0^1\frac{u^3}{1+u^4}\,du = \frac{1}{4}\ln(1+u^4)\Big]_0^1 = \frac{\ln 2}{4}.$




**3A-6** 0; $\displaystyle 2\iint_S e^x\,dA$, $S=$ right half of $R$; $\displaystyle 4\iint_Q x^2\,dA$, $Q=$ first quadrant

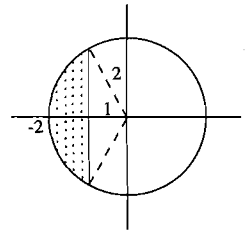
0; $\displaystyle 4\iint_Q x^2\,dA$; 0

**3A-7** a) $\displaystyle x^4+y^4\ge 0 \Rightarrow \frac{1}{1+x^4+y^4}\le 1$

b) $\displaystyle\iint_R\frac{x\,dA}{1+x^2+y^2} \le \int_0^1\int_0^1\frac{x}{1+x^2}\,dx\,dy = \frac{1}{2}\ln(1+x^2)\Big]_0^1 = \frac{\ln 2}{2} < \frac{.7}{2}.$

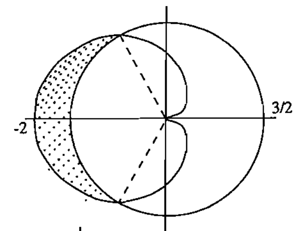
## 3B. Double Integrals in polar coordinates

### 3B-1



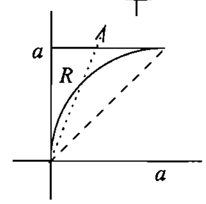

a) In polar coordinates, the line $x = -1$ becomes $r\cos\theta = -1$, or $r = -\sec\theta$. We also need the polar angle of the intersection points; since the right triangle is a 30-60-90 triangle (it has one leg 1 and hypotenuse 2), the limits are (no integrand is given):

$$\iint_R dr\,d\theta = \int_{2\pi/3}^{4\pi/3}\int_{-\sec\theta}^{2} dr\,d\theta.$$



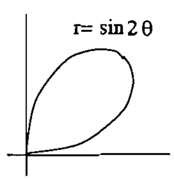

c) We need the polar angle of the intersection points. To find it, we solve the two equations $r = \frac{3}{2}$ and $r = 1 - \cos\theta$ simultanously. Eliminating $r$, we get $\frac{3}{2} = 1 - \cos\theta$, from which $\theta = 2\pi/3$ and $4\pi/3$. Thus the limits are (no integrand is given):

$$\iint_R dr\,d\theta = \int_{2\pi/3}^{4\pi/3}\int_{3/2}^{1-\cos\theta} dr\,d\theta.$$

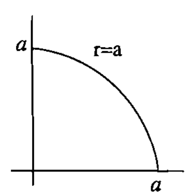



d) The circle has polar equation $r = 2a\cos\theta$. The line $y = a$ has polar equation $r\sin\theta = a$, or $r = a\csc\theta$. Thus the limits are (no integrand):

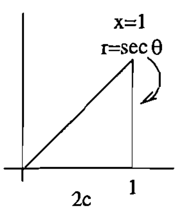
$$\iint_R dr\,d\theta = \int_{\pi/4}^{\pi/2}\int_{2a\cos\theta}^{a\csc\theta} dr\,d\theta.$$



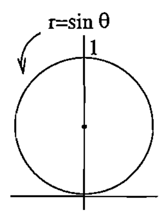
2a



2b



2c



2d

**3B-2** a) $\displaystyle\int_0^{\pi/2}\int_0^{\sin 2\theta}\frac{r\,dr\,d\theta}{r} = \int_0^{\pi/2}\sin 2\theta\,d\theta = -\frac{1}{2}\cos 2\theta\Big]_0^{\pi/2} = -\frac{1}{2}(-1-1) = 1.$

b) $\displaystyle\int_0^{\pi/2}\int_0^{a}\frac{r}{1+r^2}\,dr\,d\theta = \frac{\pi}{2}\cdot\frac{1}{2}\ln(1+r^2)\Big]_0^{a} = \frac{\pi}{4}\ln(1+a^2).$

c) $\displaystyle\int_0^{\pi/4}\int_0^{\sec\theta}\tan^2\theta\cdot r\,dr\,d\theta = \frac{1}{2}\int_0^{\pi/4}\tan^2\theta\,\sec^2\theta\,d\theta = \frac{1}{6}\tan^3\theta\Big]_0^{\pi/4} = \frac{1}{6}.$

d) $\displaystyle\int_0^{\pi/2}\int_0^{\sin\theta}\frac{r}{\sqrt{1-r^2}}\,dr\,d\theta$

Inner: $\displaystyle -\sqrt{1-r^2}\Big]_0^{\sin\theta} = 1 - \cos\theta$ Outer: $\displaystyle \theta - \sin\theta\Big]_0^{\pi/2} = \pi/2 - 1.$

**3B-3** a) the hemisphere is the graph of $z = \sqrt{a^2 - x^2 - y^2} = \sqrt{a^2 - r^2}$, so we get

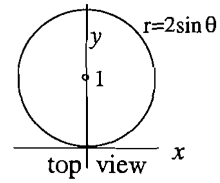
$$\iint_R \sqrt{a^2 - r^2}\,dA = \int_0^{2\pi}\int_0^{a}\sqrt{a^2-r^2}\,r\,dr\,d\theta = 2\pi\cdot -\frac{1}{3}(a^2-r^2)^{3/2}\Big]_0^{a} = 2\pi\cdot\frac{1}{3}a^3 = \frac{2}{3}\pi a^3.$$

b) $\displaystyle\int_0^{\pi/2}\int_0^a (r\cos\theta)(r\sin\theta) r\,dr\,d\theta = \int_0^a r^3\,dr \int_0^{\pi/2} \sin\theta\cos\theta\,d\theta = \frac{a^4}{4}\cdot\frac{1}{2} = \frac{a^4}{8}.$

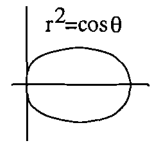
c) In order to be able to use the integral formulas at the beginning of 3B, we use symmetry about the $y$-axis to compute the volume of just the right side, and double the answer.

$$\iint_R \sqrt{x^2+y^2}\,dA = 2\int_0^{\pi/2}\int_0^{2\sin\theta} r\,r\,dr\,d\theta = 2\int_0^{\pi/2}\frac{1}{3}(2\sin\theta)^3\,d\theta$$

$= 2\cdot\dfrac{8}{3}\cdot\dfrac{2}{3} = \dfrac{32}{9}$, by the integral formula at the beginning of **3B**.




d) $\displaystyle 2\int_0^{\pi/2}\int_0^{\sqrt{\cos\theta}} r^2\,r\,dr\,d\theta = 2\int_0^{\pi/2}\frac{1}{4}\cos^2\theta\,d\theta = 2\cdot\frac{1}{4}\cdot\frac{\pi}{4} = \frac{\pi}{8}.$

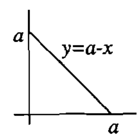



## 3C. Applications of Double Integration

**3C-1** Placing the figure so its legs are on the positive $x$- and $y$-axes,

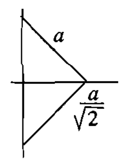
a) M.I. $= \displaystyle\int_0^a\int_0^{a-x} x^2\,dy\,dx$ Inner: $x^2y\Big]_0^{a-x} = x^2(a-x)$; Outer: $\displaystyle\frac{1}{3}x^3a - \frac{1}{4}x^4\Big]_0^a = \frac{1}{12}a^4.$

b) $\displaystyle\iint_R (x^2+y^2)\,dA = \iint_R x^2\,dA + \iint_R y^2\,dA = \frac{1}{12}a^4 + \frac{1}{12}a^4 = \frac{1}{6}a^4.$




c) Divide the triangle symmetrically into two smaller triangles, their legs are $\dfrac{a}{\sqrt{2}}$;

Using the result of part (a), M.I. of $R$ about hypotenuse $= 2\cdot\dfrac{1}{12}\left(\dfrac{a}{\sqrt{2}}\right)^4 = \dfrac{a^4}{24}$




**3C-2** In both cases, $\bar{x}$ is clear by symmetry; we only need $\bar{y}$.

a) Mass is $\displaystyle\iint_R dA = \int_0^\pi \sin x\,dx = 2$

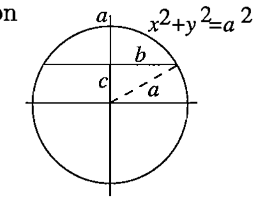
$y$-moment is $\displaystyle\iint_R y\,dA = \int_0^\pi\int_0^{\sin x} y\,dy\,dx = \frac{1}{2}\int_0^\pi \sin^2 x\,dx = \frac{\pi}{4}$; therefore $\bar{y} = \dfrac{\pi}{8}$.

b) Mass is $\displaystyle\iint_R y\,dA = \frac{\pi}{4}$, by part (a). Using the formulas at the beginning of **3B**,

$y$-moment is $\displaystyle\iint_R y^2\,dA = \int_0^\pi\int_0^{\sin x} y^2\,dy\,dx = 2\int_0^{\pi/2}\frac{\sin^3 x}{3}\,dx = 2\cdot\frac{1}{3}\cdot\frac{2}{3} = \frac{4}{9},$

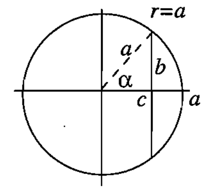
Therefore $\bar{y} = \dfrac{4}{9}\cdot\dfrac{4}{\pi} = \dfrac{16}{9\pi}.$

**3C-3** Place the segment either horizontally or vertically, so the diameter is respectively on the $x$ or $y$ axis. Find the moment of half the segment and double the answer.




(a) (Horizontally, using rectangular coordinates) Note that $a^2 - c^2 = b^2$.

$$\int_0^b \int_c^{\sqrt{a^2-x^2}} y\,dy\,dx = \int_0^b \frac{1}{2}(a^2 - x^2 - c^2)\,dx = \frac{1}{2}\left[b^2 x - \frac{x^3}{3}\right]_0^b = \frac{1}{3}b^3; \quad \text{ans: } \frac{2}{3}b^3.$$



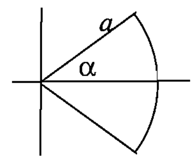

(b) (Vertically, using polar coordinates). Note that $x = c$ becomes $r = c\sec\theta$.

$$\text{Moment} = \int_0^\alpha \int_{c\sec\theta}^a (r\cos\theta)\,r\,dr\,d\theta \qquad \text{Inner: } \tfrac{1}{3}r^3\cos\theta\Big]_{c\sec\theta}^a = \tfrac{1}{3}(a^3\cos\theta - c^3\sec^2\theta)$$

$$\text{Outer: } \tfrac{1}{3}\Big[a^3\sin\theta - c^3\tan\theta\Big]_0^\alpha = \tfrac{1}{3}(a^2 b - c^2 b) = \tfrac{1}{3}b^3; \quad \text{ans: } \frac{2}{3}b^3.$$

**3C-4** Place the sector so its vertex is at the origin and its axis of symmetry lies along the positive $x$-axis. By symmetry, the center of mass lies on the $x$-axis, so we only need find $\bar{x}$.

Since $\delta = 1$, the area and mass of the disc are the same: $\pi a^2 \cdot \dfrac{2\alpha}{2\pi} = a^2\alpha$.

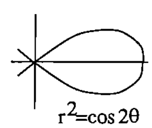



$$x\text{-moment: } 2\int_0^\alpha \int_0^a r\cos\theta \cdot r\,dr\,d\theta \qquad \text{Inner: } \tfrac{2}{3}r^3\cos\theta\Big]_0^a;$$

$$\text{Outer: } \tfrac{2}{3}a^3\sin\theta\Big]_0^\alpha = \tfrac{2}{3}a^3\sin\alpha \qquad \bar{x} = \frac{\tfrac{2}{3}a^3\sin\alpha}{a^2\alpha} = \frac{2}{3}\cdot a\cdot\frac{\sin\alpha}{\alpha}.$$

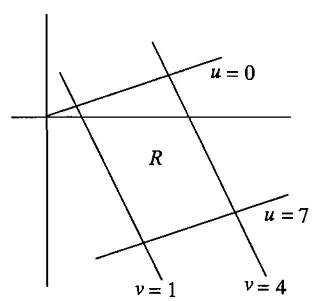
**3C-5** By symmetry, we use just the upper half of the loop and double the answer. The upper half lies between $\theta = 0$ and $\theta = \pi/4$.

$$2\int_0^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} r^2\,r\,dr\,d\theta = 2\int_0^{\pi/4}\frac{1}{4}a^4\cos^2 2\theta\,d\theta$$




$$\text{Putting } u = 2\theta, \text{ the above} = \frac{a^4}{2\cdot 2}\int_0^{\pi/2}\cos^2 u\,du = \frac{a^4}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi a^4}{16}.$$

## 3D. Changing Variables




**3D-1** Let $u = x - 3y$, $v = 2x + y$; $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -3 \\ 2 & 1 \end{vmatrix} = 7$; $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{7}$.

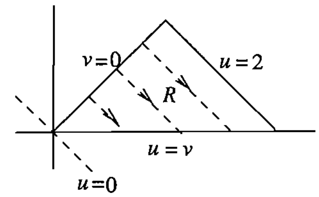
$$\iint_R \frac{x-3y}{2x+y}\,dx\,dy \;=\; \frac{1}{7}\int_0^7\int_1^4 \frac{u}{v}\,dv\,du$$

$$\text{Inner: } u\ln v\Big]_1^4 = u\ln 4; \quad \text{Outer: } \tfrac{1}{2}u^2\ln 4\Big]_0^7 = \frac{49\ln 4}{2}; \quad \text{Ans: } \frac{1}{7}\frac{49\ln 4}{2} \;=\; 7\ln 2$$

**3D-2** Let $u = x + y, \quad v = x - y$. Then $\dfrac{\partial(u,v)}{\partial(x,y)} = 2; \quad \dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{2}$.

To get the $uv$-equation of the bottom of the triangular region:

$y = 0 \quad \Rightarrow \quad u = x, \ v = x \quad \Rightarrow \quad u = v.$




$$\iint_R \cos\left(\frac{x-y}{x+y}\right) dx\,dy \;=\; \frac{1}{2}\int_0^2 \int_0^u \cos\frac{v}{u}\,dv\,du$$

Inner: $u \sin\dfrac{v}{u}\Big]_0^u = u\sin 1$ Outer: $\tfrac{1}{2}u^2 \sin 1\Big]_0^2 = 2\sin 1$ Ans: $\sin 1$

**3D-3** Let $u = x, \ v = 2y$; $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{vmatrix} = \dfrac{1}{2}$

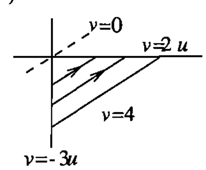
Letting $R$ be the elliptical region whose boundary is $x^2 + 4y^2 = 16$ in $xy$-coordinates, and $u^2 + v^2 = 16$ in $uv$-coordinates (a circular disc), we have

$$\iint_R (16 - x^2 - 4y^2)\,dy\,dx \;=\; \frac{1}{2}\iint_R (16 - u^2 - v^2)\,dv\,du$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^4 (16 - r^2)\,r\,dr\,d\theta \;=\; \pi\left(16\frac{r^2}{2} - \frac{r^4}{4}\right)_0^4 \;=\; 64\pi.$$

**3D-4** Let $u = x + y, \quad v = 2x - 3y$; then $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -5; \quad \dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{5}$.

We next express the boundary of the region $R$ in $uv$-coordinates.
For the $x$-axis, we have $y = 0$, so $u = x, \ v = 2x$, giving $v = 2u$.
For the $y$-axis, we have $x = 0$, so $u = y, \ v = -3y$, giving $v = -3u$.



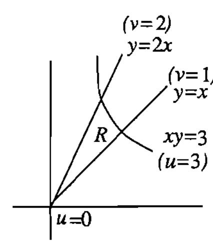

It is best to integrate first over the lines shown, $v = c$; this means $v$ is held constant, i.e., we are integrating first with respect to $u$. This gives

$$\iint_R (2x - 3y)^2 (x+y)^2 dx\,dy \;=\; \int_0^4 \int_{-v/3}^{v/2} v^2 u^2 \frac{du\,dv}{5}.$$

Inner: $\dfrac{v^2}{15}u^3\Big]_{-v/3}^{v/2} = \dfrac{v^2}{15}v^3\left(\dfrac{1}{8} - \dfrac{-1}{27}\right)$ Outer: $\dfrac{v^6}{6\cdot 15}\left(\dfrac{1}{8} + \dfrac{1}{27}\right)_0^4 = \dfrac{4^6}{6\cdot 15}\left(\dfrac{1}{8} + \dfrac{1}{27}\right)$.

**3D-5** Let $u = xy, \ v = y/x$; in the other direction this gives $y^2 = uv, \ x^2 = u/v$.

We have $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ -y/x^2 & 1/x \end{vmatrix} = \dfrac{2y}{x} = 2v; \quad \dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{2v}$; this gives




$$\iint_R (x^2 + y^2)\,dx\,dy \;=\; \int_0^3 \int_1^2 \left(\frac{u}{v} + uv\right)\frac{1}{2v}\,dv\,du.$$

Inner: $\dfrac{-u}{2v} + \dfrac{u}{2}v\Big]_1^2 = u\left(-\dfrac{1}{4} + 1 + \dfrac{1}{2} - \dfrac{1}{2}\right) = \dfrac{3u}{4}$; Outer: $\dfrac{3}{8}u^2\Big]_0^3 = \dfrac{27}{8}$.

**3D-8** a) $y = x^2$; therefore $u = x^3, \ v = x$, which gives $u = v^3$.

b) We get $\dfrac{u}{v} + uv = 1$, or $u = \dfrac{v}{v^2+1}$; (cf. 3D-5)