So -- So, yesterday we learned about the questions of planes and how to think of 3x3 linear systems in terms of intersections of planes and how to think about them geometrically. And, that in particular led us to see which cases actually we don't have a unique solution to the system, but maybe we have no solutions or infinitely many solutions because maybe the line at intersection of two of the planes happens to be parallel to the other plane. So, today, we'll start by looking at the equations of lines.

And, so in a way it seems like something which we've already seen last time because we have seen that we can think of a line as the intersection of two planes.

And, we know what equations of planes look like.

So, we could describe a line by two equations telling us about the two planes that intersect on the line.

But that's not the most convenient way to think about the line usually, though, because when you have these two questions, have you solve them?

Well, OK, you can, but it takes a bit of effort.

So, instead, there is another representation of a line. So, if you have a line in space, well, you can imagine may be that you have a point on it.

And, that point is moving in time.

And, the line is the trajectory of a point as time varies.

So, think of a line as the trajectory of a moving point.

And, so when we think of the trajectory of the moving point, that's called a parametric equation.

OK, so we are going to learn about parametric equations of lines. So, let's say for example that we are looking at the line. So, to specify a line in space, I can do that by giving you two points on the line or by giving you a point and a vector parallel to the line.

For example, so let's say I give you two points on the line: (-1,2,2), and the other point will be (1,3,-1). So, OK, it's pretty good because we have two points in that line.

Now, how do we find all the other points?

Well, the other points in between these guys and also on either side. Let's imagine that we have a point that's moving on the line, and at time zero, it's here at Q0. And, in a unit time, I'm not telling you what the unit is.

It could be a second. It could be an hour.

It could be a year. At t=1, it's going to be at Q1.

And, it moves at a constant speed.

So, maybe at time one half, it's going to be here.

Times two, it would be over there.

And, in fact, that point didn't start here.

Maybe it's always been moving on that line.

At time minus two, it was down there.

So, let's say Q(t) is a moving point, and at t=0 it's at Q0.

And, let's say that it moves. Well, we couldn't make it move in any way we want. But, probably the easiest to find, so our role is going to find formulas for a position of this moving point in terms of t. And, we'll use that to say, well, any point on the line is of this form where you have to plug in the current value of t depending on when it's hit by the moving point.

So, perhaps it's easiest to do it if we make it move at a constant speed on the line, and that speed is chosen so that at time one, it's at Q1.

So, the question we want to answer is, what is the position at time t, so, the point Q(t)?

Well, to answer that we have an easy observation, which is that the vector from Q0 to Q of t is proportional to the vector from Q0 to Q1. And, what's the proportionality factor here? Yeah, it's exactly t.

At time one, Q0 Q is exactly the same.

Maybe I should draw another picture again.

I have Q0. I have Q1, and after time t, I'm here at Q of t where this vector from Q0 Q(t) is actually going to be t times the vector Q0 Q1.

So, when t increases, it gets longer and longer.

So, does everybody see this now? Is that OK?

Any questions about that? Yes?

OK, so I will try to avoid using blue.

Thanks for, that's fine. So, OK, I will not use blue anymore. OK, well, first let me just make everything white just for now.

This is the vector from Q0 to Q(t).

This is the point Q(t). OK, is it kind of visible now?

OK, thanks for pointing it out. I will switch to brighter colors. So, OK, so apart from that, I claim now we can find the position of its moving point because, well, this vector, Q0Q1 we can find from the coordinates of Q0 and Q1.

So, we just subtract the coordinates of Q0 from those of Q1 will get that vector Q0 Q1 is OK, so, if I look at it, well, so let's call x(t), y(t), and z(t) the coordinates of the point that's moving on the line.

Then we get x of t minus, well, actually plus one equals t times two. I'm writing the components of Q0Q(t). And here, I'm writing t times Q0Q1. y(t) minus two equals t, and z(t) minus two equals -3t. So, in other terms, the more familiar way that we used to write these equations, let me do it that way instead, minus one plus 2t, y(t) = 2 t, z(t) = 2 - 3t. And, if you prefer, I can just say Q(t) is Q0 plus t times vector Q0Q1.

OK, so that's our first parametric equation of a line in this class. And, I hope you see it's not extremely hard. In fact, parametric equations of lines always look like that. x, y, and z are functions of t but are of the form a constant plus a constant times t.

The coefficients of t tell us about a vector along the line.

Here, we have a vector, Q0Q1, which is .

And, the constant terms tell us about where we are at t=0.

If I plug t=0 these guys go away, I get minus 1,2, 2. That's my starting point.

OK, so, any questions about that?

No? OK, so let's see, now, what we can do with these parametric equations.

So, one application is to think about the relative position of a line and a plane with respect to each other.

So, let's say that we take still the same line up there, and let's consider the plane with the equation x 2y 4z = 7.

OK, so I'm giving you this plane.

And, the questions that we are going to ask ourselves are, well, does the line intersect the plane?

And, where does it intersect the plane?

So, let's start with the first primary question that maybe we should try to understand. We have these points.

We have this plane, and we have these points, Q0 and Q1. I'm going to draw them in completely random places. Well, are Q0 and Q1 on the same side of a plane or on different sides, on opposite sides of the planes? Could it be that maybe one of the points is in the plane? So, I think I'm going to let you vote on that. So, is that readable?

Is it too small? OK, so anyway, the question says, relative to the plane, x 2y 4z = 7. This point, Q0 and Q1, are they on the same side, on opposite sides, is one of them on the plane, or we can't decide?

OK, that should be better. So, I see relatively few answers. OK, it looks like also a lot of you have forgotten the cards and, so I see people raising two fingers, I see people raising three fingers.

And, I see people raising four fingers.

I don't see anyone answering number one.

So, the general idea seems to be that either they are on opposite sides. Maybe one of them is on the plane. Well, let's try to see.

Is one of them on the plane? Well, let's check.

OK, so let's look at the point, sorry.

I have one blackboard to use here.

So, I take the point Q0, which is at (-1,2,2).

Well, if I plug that into the plane equation, so, x 2y 4z will equal minus one plus two times two plus four times two. That's, well, four plus eight, 12 minus one, 11. That, I think, is bigger than seven. OK, so Q0 is not in the plane.

Let's try again with Q1. (1,3, - 1) well, if we plug that into x 2y 4z, we'll have one plus two times three makes seven. But, we add four times negative one. We add up with three less than seven. Well, that one is not in the plane, either. So, I don't think, actually, that the answer should be number three.

So, let's get rid of answer number three.

OK, let's see, in light of this, are you willing to reconsider your answer?

OK, so I think now everyone seems to be interested in answering number two. And, I would agree with that answer. So, let's think about it.

These points are not in the plane, but they are not in the plane in different ways. One of them somehow overshoots; we get 11. The other one we only get 3.

That's less than seven. If you think about how a plan splits space into two half spaces on either side, well, one of them is going to be the point where x 2y 4z is less than seven. And, the other one will be, so, that's somehow this side. And, that's where Q1 is.

And, the other side is where x 2y 4z is actually bigger than seven. And, to go from one to the other, well, x 2y 4z needs to go through the value seven.

If you're moving along any path from Q0 to Q1, this thing will change continuously from 11 to 3.

At some time, it has to go through 7.

Does that make sense? So, to go from Q0 to Q1 we need to cross P at some place. So, they're on opposite sides.

OK, now that doesn't quite finish answering the question that we had, which was, where does the line intersect the plane? But, why can't we do the same thing? Now, we know not only the points Q0 and Q1, we know actually any point on the line because we have a parametric equation up there telling us, where is the point that's moving on the line at time t? So, what about the moving point, Q(t)? Well, let's plug its coordinates into the plane equation.

So, we'll take x(t) 2y(t) 4z(t). OK, that's equal to, well, (-1 2t) 2( 2 t) 4( 2 - 3t).

So, if you simplify this a bit, you get 2t 2t -12t.

That should be -8t. And, the constant term is minus one plus four plus eight is 11. OK, and we have to compare that with seven. OK, the question is, is this ever equal to seven? Well, so, Q(t) is in the plane exactly when -8t 11 equals seven.

And, that' the same. If you manipulate this, you will get t equals one half. In fact, that's not very surprising. If you look at these values, 11 and three, you see that seven is actually right in between. It's the average of these two numbers. So, it would make sense that it's halfway in between Q0 and Q1, but we will get seven.

OK, and that at that time, Q at time one half, well, let's plug the values. So, minus one plus 2t will be zero. Two plus t will be two and a half of five halves, and two minus three halves will be one half, OK? So, this is where the line intersects the plane.

So, you see that's actually a pretty easy way of finding where a line on the plane intersects each other.

If we can find a parametric equation of a line and an equation of a plane, but we basically just plug one into the other, and see at what time the moving point hits the plane so that we know where this.

OK, other questions about this? Yes?

Sorry, can you say that? Yes, so what if we don't get a solution? What happens?

So, indeed our line could have been parallel to the plane or maybe even contained in the plane.

Well, if the line is parallel to the plane then maybe what happens is that what we plug in the positions of the moving point, we actually get something that never equals seven because maybe we get actually a constant.

Say that we had gotten, I don't know, 13 all the time. Well, when is 13 equal to seven?

The answer is never. OK, so that's what would tell you that the line is actually parallel to the plane.

You would not find a solution to the equation that you get at the end. Yes?

So, if there's no solution at all to the equation that you get, it means that at no time is the traveling point going to be in the plane. That means the line really does not have the plane ever. So, it has to be parallel outside of it. On the other hand, if a line is inside the plane, then that means that no matter what time you choose, you always get seven.

OK, that's what would happen if a line is in the plane.

You always get seven. So, maybe I should write this down. So, if a line is in the plane then plugging x(t), y(t), z(t) into the equation, we always get, well, here in this case seven or whatever the value should be for the plane, If the line is parallel to the plane -- -- in fact, we, well, get, let's see, another constant.

So, in fact, you know, when you plug in these things, normally you get a quantity that's of a form, something times t plus a constant because that's what you plug into the equation of a plane. And so, in general, you have an equation of the form, something times t plus something equals something. And, that usually has a single solution. And, the special case is if this coefficient of t turns out to be zero in the end, and that's actually going to happen, exactly when the line is either parallel or in the plane.

In fact, if you think this through carefully, the coefficient of t that you get here, see, it's one times two plus two times one plus four times minus three. It's the dot product between the normal vector of a plane and the vector along the line.

So, see, this coefficient becomes zero exactly when the line is perpendicular to the normal vector.

That means it's parallel to the plane.

So, everything makes sense. OK, if you're confused about what I just said, you can ignore it.

OK, more questions? No? OK, so if not, let's move on to linear parametric equations.

So, I hope you've seen here that parametric equations are a great way to think about lines. There are also a great way to think about actually any curve, any trajectory that can be traced by a moving point. So -- -- more generally, we can use parametric equations -- -- for arbitrary motions -- -- in the plane or in space. So, let's look at an example.

Let's take, so, it's a famous curve called a cycloid. A cycloid is something that you can actually see sometimes at night when people are biking If you have something that reflects light on the wheel.

So, let me explain what's the definition of a cycloid.

So, I should say, I've seen a lecture where, actually, the professor had a volunteer on a unicycle to demonstrate how that works. But, I didn't arrange for that, so instead I will explain it to you using more conventional means. So, let's say that we have a wheel that's rolling on a horizontal ground.

And, as it's rolling of course it's going to turn.

So, it's going to move forward to a new position.

And, now, let's mention that we have a point that's been painted red on the circumference of the wheel.

And, initially, that point is here.

So, as the wheel stops rotating, well, of course, it moves forward, and so it turns on itself.

So, that point starts falling back behind the point of contact because the wheel is rotating at the same time as it's moving forward. And so, the cycloid is the trajectory of this moving point. OK, so the cycloid is obtained by considering, so we have a wheel, let's say, of radius a. So, this height here is (a) rolling on the floor which is the x axis.

And, let's -- And, we have a point, P, that's painted on the wheel. Initially, it's at the origin.

But, of course, as time goes by, it moves on the wheel. P is a point on the rim of the wheel, and it starts at the origin.

So, the question is, what happens?

In particular, can we find the position of this point, x(t), y(t), as a function of time?

So, that's the reason why I have this computer.

So, I'm not sure it will be very easy to visualize, but so we have a wheel, well, I hope you can vaguely see that there's a circle that's moving.

The wheel is green here. And, there's a radius that's been painted blue in it. And, that radius rotates around the wheel as the wheel is moving forward.

So, now, let's try to paint, actually, the trajectory of a point. [LAUGHTER] OK, so that's what the cycloid looks like.

[APPLAUSE] OK, so -- So the cycloid, well, I guess it doesn't quite look like what I've drawn.

It looks like it goes a bit higher up, which will be the trajectory of this red point. And, see, it hits the bottom once in a while. It forms these arches because when the wheel has rotated by a full turn, then you're basically back at the same situation, except a bit further along the route.

So, if we do it once more, you see the point now is at the top, and now it's at the bottom. And then we start again.

It's at the top, and then again at the bottom.

OK. No.

[LAUGHTER] OK, so the question that we want to answer is what is the position x(t), y(t), of the point P? OK, so actually, I'm writing x(t), y(t).

That means that I have, maybe I'm expressing the position in terms of time. Let's see, is time going to be a good thing to do? Well, suddenly, the position changes over time. But doesn't actually matter how fast the wheel is rolling? No, because I can just play the motion fast-forward. The wheel will be going faster, but the trajectory is still the same.

So, in fact, time is not the most relevant thing here. What matters to us now is how far the wheel has gone. So, we could use as a parameter, for example, the distance by which the wheel has moved. We can do even better because we see that, really, the most complicated thing that happens here is really the rotation.

So, maybe we can actually use the angle by which the wheel has turned to parameterize the motion.

So, there's various choices. You can choose whichever one you prefer. But, I think here, we will get the simplest answer if we parameterize things by the angle. So, in fact, instead of t I will be using what's called theta as a function of the angle, theta, by which the wheel has rotated. So, how are we going to do that?

Well, because we are going to try to use our new knowledge, let's try to do it using vectors in a smart way.

So, let me draw a picture of the wheel after things have rotated by a certain amount. So, maybe my point, P, now, is here. And, so the wheel has rotated by this angle here. And, I want to find the position of my point, P, OK?

So, the position of this point, P, is going to be the same as knowing the vector OP from the origin to this moving point.

So, I haven't really simplify the problem yet because we don't really know about vector OP. But, maybe we know about simpler vectors where some will be OP.

So, let's see, let's give names to a few of our points. For example, let's say that this will be point A.

A is the point where the wheel is touching the road.

And, B will be the center of the wheel.

Then, it looks like maybe I have actually a chance of understanding vectors like maybe OA doesn't look quite so scary, or AB doesn't look too bad. BP doesn't look too bad.

And, if I sum them together, I will obtain OP.

So, let's do that. So, now we've greatly simplified the problem. We had to find one vector that we didn't know. Now we have to find three vectors which we don't know. But, you will see each of them as fairly easy to think about. So, let's see.

Should we start with vector OA, maybe?

So, OA has two components. One of them should be very easy.

Well, the y component is just going to be zero, OK? It's directed along the x axis.

What about the x component? So, OA is the distance by which the wheel has traveled to get to its current position.

Yeah. I hear a lot of people saying R theta. Let me actually say a(theta) because I've called a the radius of the wheel.

So, this distance is a(theta). Why is it a(theta)?

Well, that's because the wheel, well, there's an assumption which is that the wheel is rolling on something normal like a road, and not on, maybe, ice, or something like that. S So, it's rolling without slipping. So, that means that this distance on the road is actually equal to the distance here on the circumference of the wheel. This point, P, was there, and the amount by which the things have moved can be measured either here or here. These are the same distances.

OK, so, that makes it a(theta), and maybe I should justify by saying amount by which the wheel has rolled, has moved, is equal to the, so, the distance from O to A is equal to the arc length on the circumference of the circle from A to P. And, you know that if you have a sector corresponding to an angle, theta, then its length is a times theta, provided that, of course, you express the angel in radians.

That's the reason why we always used radians in math.

Now, let's think about vector AB and vector BP.

OK, so AB is pretty easy, right, because it's pointing straight up, and its length is a.

So, it's just zero, a. Now, the most serious one we've kept for the end. What about vector BP?

So, vector BP, we know two things about it.

We know actually its length, so, the magnitude of BP -- -- a. And, we know it makes an angle, theta, with the vertical. So, that should let us find its components. Let's draw a closer picture.

Now, in the picture I'm going to center things at B.

So, I have my point P. Here I have theta.

This length is A. Well, what are the components of BP? Well, the X component is going to be? Almost.

I hear people saying things about a, but I agree with a.

I hear some cosines. I hear some sines.

I think it's actually the sine. Yes.

It's a(sin(theta)), except it's going to the left.

So, actually it will have a negative a(sin(theta)).

And, the vertical component, well, it will be a(cos(theta)), but also negative because we are going downwards.

So, it's negative a(cos(theta)). So, now we can answer the initial question because vector OP, well, we just add up OA, AB, and BP. So, the X component will be a(theta) - a(sin(theta)). And, a-a(cos(theta)).

OK. So, any questions about that?

OK, so, what's the answer? Because this thing here is the x coordinate as a function of theta, and that one is the y coordinate as a function of theta.

So, now, just to show you that we can do a lot of things when we have a parametric equation, here is a small mystery.

So, what happens exactly near the bottom point?

What does the curve look like? The computer tells us, well, it looks like it has some sort of pointy thing, but isn't that something of a display?

Is it actually what happens? So, what do you think happens near the bottom point? Remember, we had that picture.

Let me show you once more, where you have these corner-like things at the bottom.

Well, actually, is it indeed a corner with some angle between the two directions?

Does it make an angle? Or, is it actually a smooth curve without any corner, but we don't see it because it's too small to be visible on the computer screen?

Does it actually make a loop? Does it actually come down and then back up without going to the left or to the right and without making an angle? So, yeah, I see the majority votes for answers number two or four.

And, well, at this point, we can't quite tell.

So, let's try to figure it out from these formulas.

The way to answer that for sure is to actually look at the formulas. OK, so question that we are trying to answer now is what happens near the bottom point?

OK, so how do we answer that? Well, we should probably try to find simpler formulas for these things.

Well, to simplify, let's divide everything by a.

Let's rescale everything by a. If you want, let's say that we take the unit of length to be the radius of our wheel. So, instead of measuring things in feet or meters, we'll just measure them in radius. So, take the length unit to be equal to the radius. So, that means we'll have a=1.

Then, our formulas are slightly simpler.

We get x(theta) is theta - sin(theta), and y equals 1 - cos (theta). OK, so, if we want to understand what these things look like, maybe we should try to take some approximation. OK, so what about approximations? Well, probably you know that if I take the sine of a very small angle, it's close to the actual angle itself if theta is very small.

And, you know that the cosine of an angle that's very small is close to one. Well, that's pretty good.

If we use that, we will get theta minus theta, one minus one, it looks like it's not precise enough. We just get zero and zero.

That's not telling us much about what happens.

OK, so we need actually better approximations than that.

So -- So, hopefully you have seen in one variable calculus something called Taylor expansion.

That's [GROANS]. I see that -- OK, so if you have not seen Taylor expansion, or somehow it was so traumatic that you've blocked it out of your memory, let me just remind you that Taylor expansion is a way to get a better approximation than just looking at the function, its derivative.

So -- And, here's an example of where it actually comes in handy in real life. So, Taylor approximation says that if t is small, then the value of the function, f(t), is approximately equal to, well, our first guess, of course, would be f(0).

That's our first approximation. If we want to be a bit more precise, we know that when we change by t, well, t times the derivative comes in, that's for linear approximation to how the function changes.

Now, if we want to be even more precise, there's another term, which is t^2 over two times the second derivative.

And, if we want to be even more precise, you will have t^3 over six times the third derivative at zero.

OK, and you can continue, and so on.

But, we won't need more. So, if you use this here, it tells you that the sine of a smaller angle, theta, well, yeah, it looks like theta.

But, if we want to be more precise, then we should add minus theta cubed over six. And, cosine of theta, well, it's not quite one. It's close to one minus theta squared over two. OK, so these are slightly better approximations of sine and cosine for small angles.

So, now, if we try to figure out, again, what happens to our x of theta, well, it would be, sorry, theta minus theta cubed over six.

That's theta cubed over six. And y, on the other hand, is going to be one minus that. That's about theta squared over two. So, now, which one of them is bigger when theta is small? Yeah, y is much larger.

OK, if you take the cube of a very small number, it becomes very, very, very small.

So, in fact, we can look at that.

So, x, an absolute value, is much smaller than y.

And, in fact, what we can do is we can look at the ratio between y and x. That tells us the slope with which we approach the origin. So, y over x is, well, let's take the ratio of this, too.

That gives us three divided by theta.

That tends to infinity when theta approaches zero.

So, that means that the slope of our curve, the origin is actually infinite.

And so, the curve picture is really something like this.

So, the instantaneous motion, if you had to describe what happens very, very close to the origin is that your point is actually not moving to the left or to the right along with the wheel. It's moving down and up.

I mean, at the same time it is actually moving a little bit forward at the same time. But, the dominant motion, near the origin is really where it goes down and back up, so answer number four, you have vertical tangent.

OK, I think I'm at the end of time.

So, have a nice weekend. And, I'll see you on Tuesday.

So, on Tuesday I will have practice exams for next week's test.