These are the solutions to Exam 2 of **18.02**, Spring 2006.

Notice that these solutions contain some explanations

in square parentheses [. . .] that were not required.

Solutions

NAME

e-mail address(*)

code

(*) *By providing your e-mail address you are giving permission to your recitation instructor to e-mail you if did not pass the exam.*

Please mark the box next to your recitation session.

Recitation:	☐ Rec.#1	☐ Rec.#7
	☐ Rec.#2	☐ Rec.#8
	☐ Rec.#3	☐ Rec.#9
	☐ Rec.#4	☐ Rec.#10
	☐ Rec.#6	☐ Rec.#11

18.02 Exam 2 Thursday, Mar 16th, 2006

Directions: Do all the work on these pages; use reverse side if needed. Answers without accompanying reasoning may only receive partial credit. No books, notes, or calculators. Please stop when asked to and don't talk until your paper is handed in.

GRADING	
1.	/ 20
2.	/ 15
3.	/ 25
4.	/ 20
5.	/ 10
6.	/ 10
TOTAL	/ 100

Problem 1

Let $f(x, y) = xy^2 - 8y$.

a) *(5)* Find ∇f at $(2, 3)$.

$$\nabla f = \langle y^2, 2xy - 8 \rangle$$

$$\nabla f(2,3) = \langle 3^2, 2\cdot 2\cdot 3 - 8 \rangle = \langle 9, 4 \rangle$$

b) *(5)* Write the equation for the tangent plane to the graph of f through the point $(2, 3, -6)$.

$$z - (-6) = 9\cdot(x-2) + 4(y-3)$$

$$z + 6 = 9x - 18 + 4y - 12$$

Equ.: $9x + 4y - z = 36$

c) *(5)* Use a linear approximation to approximate the value $f(2.1, 2.9)$.

$$f(2.1, 2.9) \approx f(2,3) + \frac{\partial f}{\partial x}(2,3)\cdot(0.1) + \frac{\partial f}{\partial y}(2,3)\cdot(-0.1) =$$

$$= -6 + 9\cdot(0.1) + 4\cdot(-0.1) = -5.5$$

d) *(5)* Find the directional derivative of f at $(2, 3)$ in the direction $2\hat{\imath} + \hat{\jmath}$.

$$\hat{u} = \frac{2\hat{\imath} + \hat{\jmath}}{\sqrt{2^2 + 1^2}} = \frac{2\hat{\imath} + \hat{\jmath}}{\sqrt{5}} = \text{direction of } 2\hat{\imath} + \hat{\jmath}.$$

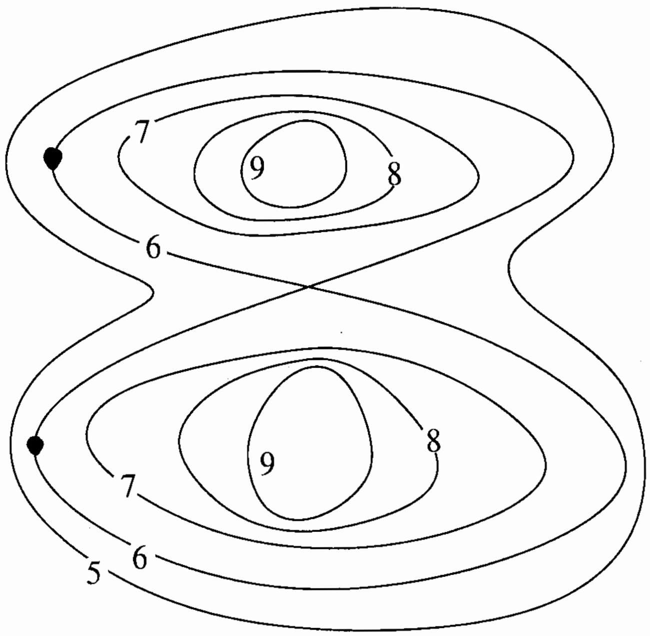
$$\left.\frac{df}{ds}\right|_{\hat{u}}(2,3) = \nabla f(2,3)\cdot\hat{u} = \langle 9, 4 \rangle \cdot \frac{\langle 2, 1 \rangle}{\sqrt{5}} =$$

$$= \frac{18 + 4}{\sqrt{5}} = \frac{22}{\sqrt{5}}$$

Problem 2

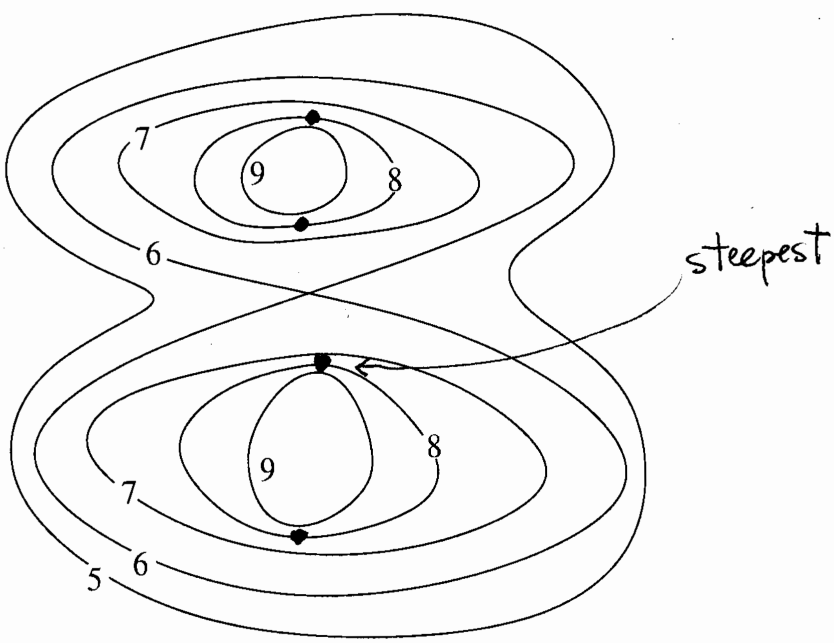
a) *(10)* On the contour plot below, mark the points of the level curve $f(x, y) = 6$ at which $f_x > 0$ and $f_y = 0$.

[Places where $\nabla f = \langle f_x, f_y \rangle$ points ⟶]



b). *(5)* On the contour plot below, mark the points of the level curve $f(x, y) = 8$ at which the slope of steepest ($|\nabla f|$ is largest).

[Any of these four points get full credit.]



Problem 3

Let $f(x,y) = x^2 + xy + y^2 + 3x$.

a) *(10)* Find and classify the critical points of f.

$\nabla f = \langle 2x+y+3,\ x+2y \rangle$

Critical points: $\nabla f = 0$

$\begin{cases} 2x+y+3=0 \\ x+2y=0 \end{cases}$

$\begin{cases} x=-2y \\ 2(-2y)+y+3=0 \end{cases}$

$\begin{cases} y=1 \\ x=-2 \end{cases}$

One critical point $P=(-2,1)$.

To classify it, use second derivative test:

$f_{xx}=2, \quad f_{xy}=1, \quad f_{yy}=2$

$\Delta = 2\cdot 2 - 1^2 = 3$

$\left.\begin{array}{l} \Delta(P) = 3 > 0 \\ f_{xx}(P) = 2 > 0 \end{array}\right\} \Rightarrow$ P is a local minimum.

b) *(10)* Find the minimum and maximum values of f in the plane. Justify your answer.

$\boxed{\text{max. value} = \text{"}+\infty\text{"}}$

For example, along the x-axis

$f(x,0) = x^2+3x \longrightarrow \infty$ as $x \to \infty$.

$\boxed{\text{Min. value} = -3}$ (attained at $(-2,1)$)

To justify, must be sure there are no smaller values than -3 as $x \to \pm\infty$ and/or $y \to \pm\infty$.

$\boxed{\text{Reason}}$: Completing the square

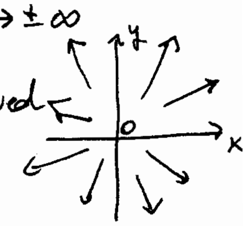
$x^2+xy+y^2 = \left(x+\frac{1}{2}y\right)^2 + \frac{3}{4}y^2$, so

$f \to +\infty$ as x and/or $y \to \pm\infty$

in any direction.

Since the min cannot be achieved as $x \to \pm\infty$ and/or $y \to \pm\infty$,

it must be achieved at the critical point $(-2,1)$ as claimed above.



c) *(5)* Find the minimum and maximum values of f in the region $x \geq 1$.

$\boxed{\text{max. value} = \text{"}+\infty\text{"}}$

(Same reason as above in (b).)

The minimum is attained at the boundary $x=1$, because there are no critical points in the domain and $f \to +\infty$ as $x \to +\infty$ and/or $y \to \pm\infty$

(same reason as in (b).)

$f(1,y) = y^2+y+4$, so the minimum is attained at $\left(1,-\frac{1}{2}\right)$.

$f\left(1,-\frac{1}{2}\right) = \frac{1}{4} - \frac{1}{2} + 4 = \frac{15}{4}$.

$\boxed{\text{Min. value} = \frac{15}{4}}$

Problem 4

Suppose that $u = x + y^2$, $v = xy^{-2}$.

a) *(10)* Express the derivatives w_x and w_y in terms of w_u, w_v (and x and y).

$$W_x = W_u \cdot u_x + W_v \cdot v_x$$
$$W_y = W_u \cdot u_y + W_v \cdot v_y$$

$$u_x = 1 \qquad v_x = y^{-2}$$
$$u_y = 2y \qquad v_y = -2xy^{-3}$$

Hence,
$$W_x = W_u + y^{-2} \cdot W_v.$$
$$W_y = 2y \cdot W_u + (-2xy^{-3}) \cdot W_v.$$

b) *(10)* Express $xw_x + \frac{1}{2}yw_y$ in terms of w_u and w_v. Write the coefficients as functions of u and v.

$$xW_x + \frac{1}{2}y \cdot W_y = x \cdot W_u + xy^{-2} W_v + y^2 W_u - xy^{-2} W_v =$$
$$= x \cdot W_u + y^2 W_u = u \cdot W_u.$$

Problem 5

(10) Set up (but do not solve) Lagrange multiplier equations for the point of the surface $2x^3 - yz^2 + xyz = 4$ closest to the origin.

We want to minimize $f(x,y,z) = x^2+y^2+z^2$, that is the square of the distance from the origin, subject to the constraint $g(x,y,z)=0$ where $g(x,y,z) = 2x^3 - yz^2 + xyz - 4$.

Lagrange multiplier equations:

$$\begin{cases} \nabla f = \lambda \cdot \nabla g \\ g = 0 \end{cases}$$

$$\nabla f = \langle 2x, 2y, 2z \rangle$$
$$\nabla g = \langle 6x^2 + yz, -z^2 + xz, -2yz + xy \rangle$$

$$\begin{cases} 2x = \lambda(6x^2 + yz) \\ 2y = \lambda(-z^2 + xz) \\ 2z = \lambda(-2yz + xy) \\ 2x^3 - yz^2 + xyz - 4 = 0 \end{cases}$$

Problem 6

Suppose that $f(x,y,z)$ is a function satisfying $\nabla f = 2\hat{\imath}+3\hat{\jmath}+\hat{k}$ at $(7,-8,1)$ and that $z = z(x,y)$ is the root of the cubic equation $z^3+xz+y=0$. There is only one root z if $x>0$ and, in particular, at $(x,y)=(7,-8)$, $z=1$.

(10) Let $g(x,y) = f(x,y,z(x,y))$; find ∇g at $(x,y)=(7,-8)$.

$$dg = df = f_x\,dx + f_y\,dy + f_z\,dz =$$
$$= 2dx + 3dy + dz \quad \text{at} \quad (7,-8,1).$$

Differentiating $x^3+xz+y=0$,

$$3z^2 dz + x\,dz + z\,dx + dy = 0.$$

At $(7,-8,1)$, $\quad 3\cdot(1)^2 dz + 7dz + 1\cdot dx + dy = 0$,

or $\quad dz = -\frac{1}{10}(dx+dy)$.

Hence, $\quad dg = 2dx + 3dy - \frac{1}{10}(dx+dy)$

and $\quad \nabla g(7,-8) = \langle 2-\frac{1}{10},\ 3-\frac{1}{10}\rangle = \langle 1.9,\ 2.9\rangle.$