18.02 Problem Set 4 - Solutions of Part B

Problem 1

a) The critical points are $P = \left(\frac{1}{3}, \frac{2}{3}\right)$ and $Q = \left(\frac{3}{2}, \frac{5}{4}\right)$.

In fact $\frac{\partial f}{\partial x} = 3x^2 - 6x + y + 1$ and $\frac{\partial f}{\partial y} = x - 2y + 1$.
The critical points are obtained setting $f_x = f_y = 0$.
$\frac{\partial f}{\partial y} = 0$ gives $y = \frac{x+1}{2}$.
Substituting inside $\frac{\partial f}{\partial x} = 0$ we get $6x^2 - 11x + 3 = 0$, which has two solutions: $x = 1/3; 3/2$.
Hence we get two critical points: $P = (1/3, 2/3)$ and $Q = (3/2, 5/4)$, which belong both to the square S.

b) The points of S where f attains its maximum or its minimum can be either the critical points found in (a) or points in the boundary of S.

c) The maximum of f is $\frac{13}{27}$, attained at $P = \left(\frac{1}{3}, \frac{2}{3}\right)$.
The minimum of f is $-1 - \frac{4}{3}\sqrt{\frac{2}{3}}$, attained at $\left(1 + \sqrt{\frac{2}{3}}, 0\right)$.

Evaluating f at the critical points, we find $f(P) = f(1/3, 2/3) = 13/27$ and $f(Q) = f(3/2, 5/4) = -5/16$.

Now we analyze what happens at the boundary.

$\boxed{y = 0}$ In this case we want to find minimum and maximum value of the function $f(x, 0) = x^3 - 3x^2 + x$ with $0 \le x \le 2$.
Its values at the extremal points are $f(0, 0) = 0$ and $f(2, 0) = -2$.
$\frac{\mathrm{d}f(x,0)}{\mathrm{d}x} = 3x^2 - 6x + 1$, which vanishes at $1 \pm \sqrt{\frac{2}{3}}$.
Substituting we get $f\left(1 - \sqrt{\frac{2}{3}}, 0\right) = -1 + \frac{4}{3}\sqrt{\frac{2}{3}}$ and $f\left(1 + \sqrt{\frac{2}{3}}, 0\right) = -1 - \frac{4}{3}\sqrt{\frac{2}{3}}$.

$\boxed{y = 2}$ In this case we have to analyze $f(x, 2) = x^3 - 3x^2 + 3x - 2 = (x-1)^3 - 1$, with $0 \le x \le 2$.

It is immediate to realize that the minimum can only be at $f(0,2) = -2$ and the maximum at $f(2,2) = 0$.

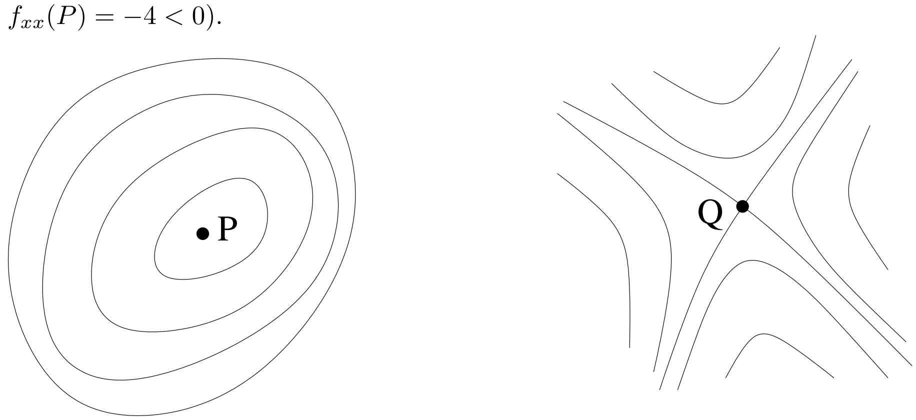
$\boxed{x = 0}$ In this case we have to analyze $f(0, y) = -y^2 + y = -y(y-1)$ with $0 \leq y \leq 2$.
Hence the maximum can only be at $f(0, 1/2) = 1/4$ and the minimum at $f(0,2) = -2$.

$\boxed{x = 2}$ In this case we have to analyze $f(2, y) = -y^2 + 3y - 2 = -(y-1)(y-2)$ with $0 \leq y \leq 2$.
Hence the minimum can only be at $f(2,0) = -2$ and the maximum at $f(2, 3/2) = 1/4$.

Comparing the values that f attains at the previous points, we get our result.

d) The point $P = \left(\frac{1}{3}, \frac{2}{3}\right)$ is a local maximum; the point $Q = \left(\frac{3}{2}, \frac{5}{4}\right)$ is a saddle point.

In fact $f_{xx} = 6x - 6$, $f_{xy} = f_{yx} = 1$ and $f_{yy} = -2$.
Hence the discriminant is $\Delta(x, y) = f_{xx}f_{yy} - f_{xy}^2 = 11 - 12x$.
$\Delta(Q) = -7 < 0$, so that Q is a saddle point.
$\Delta(P) = 7 > 0$, so that P is either a local maximum or a local minimum. In fact P is a local maximum because $f_{yy}(P) = -2 < 0$ (and equivalently $f_{xx}(P) = -4 < 0$).

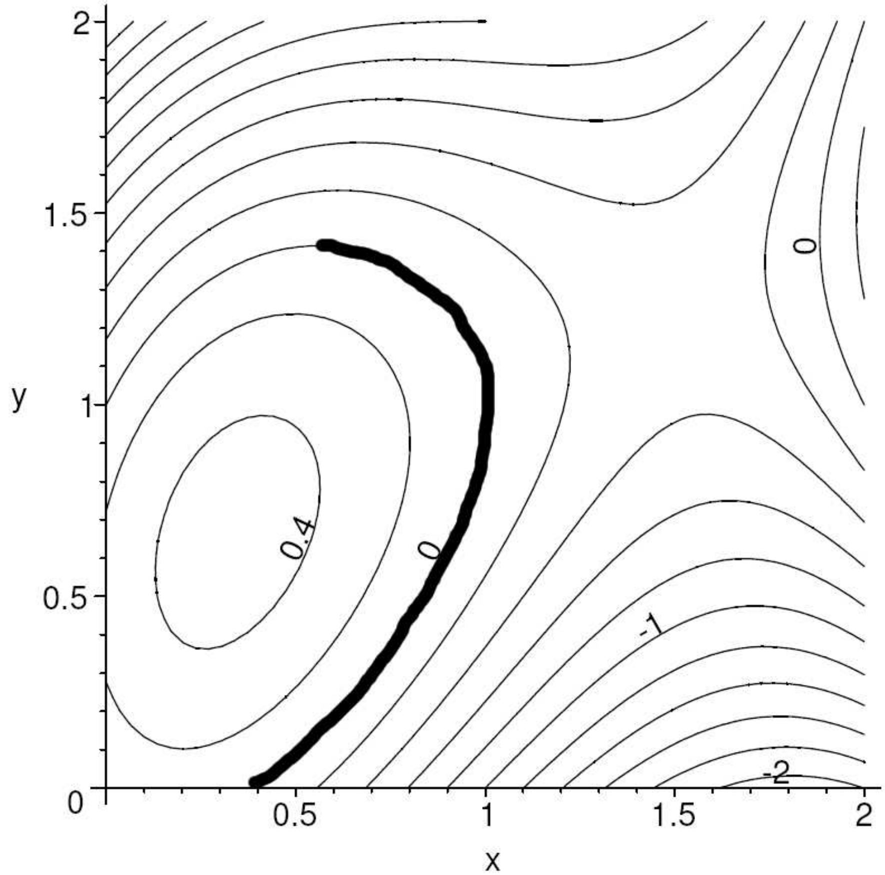


The picture above represents a sketch of the level curves of f around the critical points P and Q.

e) After (d), we can only say that $(3/2, 5/4)$ is neither a local minimum nor a local maximum and so the minimum of f is attained at the boundary. Hence we could have avoided to evaluate f at $(3/2, 5/4)$.

Problem 2

a) At the point $(1, 1/2)$ we have $f_x(1, 1/2) < 0$ and $f_y(1, 1/2) > 0$.
At the point $(1, 1)$ we have $f_x(1, 1) < 0$ and $f_y(1, 1) = 0$.



The part of the contour line through $(1, 1)$ on which $f_x < 0$ is the one emphasized in the picture above.

b) At $(1, 1/2)$ we have $f_x(1, 1/2) = -3/2$ and $f_y(1, 1/2) = 1$.
At $(1, 1)$ we have $f_x(1, 1) = -1$ and $f_y(1, 1) = 0$.

It is sufficient to plug $(1, 1/2)$ and $(1, 1)$ inside $f_x(x, y) = 3x^2 - 6x + y - 1$ and $f_y(x, y) = x - 2y + 1$ obtained in (a).

c) Using "Level curves".

The saddle is approximately at $(1.5, 1.2)$ and $f(1.5, 1.2) \approx -0.32$.
The maximum is attained approximately at $(0.4, 0.7)$ and $f(0.4, 0.7) \approx 0.48$.
The minimum is attained approximately at $(1.8, 0)$ and $f(0.8, 0) \approx -2.09$.

d) Using "Partial derivatives".
The maximum is approximately at $(0.34, 0.67)$.
The level curve through it reduces to a point.
If I move in any direction, then the value of f decreases (it's a maximum!).

The saddle is approximately at $(1.5, 1.25)$. The level curve through it is made of two branches that meet transversely.
If I move towards East or West, then the value of f increases.
If I move towards North or South, then the value of f decreases.

Problem 3

a) $\mathrm{d}R = \rho \left(\frac{\mathrm{d}L}{S} - \frac{L\mathrm{d}S}{S^2} \right) = R \left(\frac{\mathrm{d}L}{L} - \frac{\mathrm{d}S}{S} \right)$.

b) Using linear approximation, the new resistance R' is $R' \approx 1.425$ ohm.
The exact calculation gives $R' = 63/44 \approx 1.4318$ ohm.

In fact the first order approximation gives $\Delta R \approx R \left(\frac{5}{100} - \frac{0.1}{1} \right) = -0.075$, so that $R' = R + \Delta R \approx 1.5 - 0.075 = 1.425$.
To exactly compute the resistance, we notice that $\rho = RS/L$. Hence $R' = \rho \frac{L'}{S'} = R \frac{L'}{L} \frac{S}{S'} = 1.5 \frac{105}{100} \frac{1}{1.1} = \frac{63}{44} \approx 1.4318$.

Problem 4

a) $\begin{pmatrix} w_r \\ w_\theta \end{pmatrix} = \begin{pmatrix} x_r & y_r \\ x_\theta & y_\theta \end{pmatrix} \begin{pmatrix} w_x \\ w_y \end{pmatrix}$, so $A = \begin{pmatrix} x_r & y_r \\ x_\theta & y_\theta \end{pmatrix}$.

Just put

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}\\ \frac{\partial w}{\partial \theta} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta}\end{aligned}$$

in matrix form.

b) $A = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{pmatrix}$.

Just differentiate $x(r,\theta) = r\cos(\theta)$ and $y(r,\theta) = r\sin(\theta)$ with respect to r and θ and use (a).

c) $\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} r_x & \theta_x \\ r_y & \theta_y \end{pmatrix} \begin{pmatrix} u_r \\ u_\theta \end{pmatrix} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & -\frac{y}{x^2+y^2} \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{x}{x^2+y^2} \end{pmatrix} \begin{pmatrix} u_r \\ u_\theta \end{pmatrix}$.

So $B = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & -\frac{y}{x^2+y^2} \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{x}{x^2+y^2} \end{pmatrix}$.

d) $A^{-1} = \begin{pmatrix} \cos(\theta) & -\frac{1}{r}\sin(\theta) \\ \sin(\theta) & \frac{1}{r}\cos(\theta) \end{pmatrix} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & -\frac{y}{x^2+y^2} \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{x}{x^2+y^2} \end{pmatrix} = B.$

Straightforward computation, using that $\det(A) = r = \sqrt{x^2+y^2}$.

e) At $r = 5, \theta = -\pi/2$ we have $u_x = 4$ and $u_y = -1$.

In fact, the point $r = 5, \theta = -\pi/2$ has Cartesian coordinates $x = 0, y = -5$. Hence $u_x = \frac{x}{r}u_r + \left(-\frac{y}{r^2}\right)u_\theta = -\left(\frac{-5}{5^2}\right)20 = 4$ and $u_y = \frac{y}{r}u_r + \frac{x}{r^2}u_\theta = \frac{-5}{5}1 = -1$.

Problem 5

a) Maximum of $\left.\frac{\mathrm{d}f}{\mathrm{d}s}\right|_{\hat{\mathbf{u}}}(1, 1/2) = \frac{\sqrt{13}}{2}$.

Minimum of $\left.\frac{\mathrm{d}f}{\mathrm{d}s}\right|_{\hat{\mathbf{u}}}(1,1/2) = -\frac{\sqrt{13}}{2}$.

In fact, the maximum of the directional derivative is attained in direction of the gradient (and the minimum in the opposite direction). Hence, to achieve the maximum we have to set $\hat{\mathbf{u}} = \frac{\nabla f(1,1/2)}{|\nabla f(1,1/2)|}$. Moreover $\left.\frac{\mathrm{d}f}{\mathrm{d}s}\right|_{\hat{\mathbf{u}}} = \nabla f \cdot \hat{\mathbf{u}}$.
As a consequence the maximum is $\left.\frac{\mathrm{d}f}{\mathrm{d}s}\right|_{\hat{\mathbf{u}}} = \nabla f \cdot \frac{\nabla f}{|\nabla f|} = |\nabla f|$. Evaluating the gradient at $(1,1/2)$ we get $\nabla f(1,1/2) = \langle -3/2, 1\rangle$ and $|\nabla f(1,1/2)| = \frac{\sqrt{13}}{2}$.
Hence the minimum of the directional derivative is $-|\nabla f(1,1/2)|$.

b) The maximum occurs at $\hat{\mathbf{u}} = \frac{\langle -3,2\rangle}{\sqrt{13}}$ and
the minimum occurs at $\hat{\mathbf{u}} = \frac{\langle 3,-2\rangle}{\sqrt{13}}$.

As discussed in (a), the maximum is achieved at $\hat{\mathbf{u}} = \frac{\nabla f(1,1/2)}{|\nabla f(1,1/2)|}$ and the minimum at $\hat{\mathbf{u}} = -\frac{\nabla f(1,1/2)}{|\nabla f(1,1/2)|}$.

c) The directional derivative is zero for $\hat{\mathbf{u}} = \pm\frac{\langle 2,3\rangle}{\sqrt{13}}$.

In fact the directional derivative vanishes in directions perpendicular to the gradient (and so tangential to the level curves). Hence it is sufficient to rotate the direction of the gradient of $\pm\pi/2$.

d) We have used the point $(x,y) = (1.01, 0.5)$.

The maximum of the directional derivative is approximately 1.8 and it is attained at $\theta = 145^o$ or $\theta = 146^o$ or $\theta = 148^o$.
The yellow half-line (direction $\hat{\mathbf{u}}$) points in the same direction as the purple half-line (the gradient) and both are perpendicular to the blue level curve passing through the point.

The minimum is approximately -1.8 and it is attained at $\theta = 324^o$ or $\theta = 326^o$.
The yellow half-line is in the opposite direction than the purple one, and both are perpendicular to the blue level curve passing through the point.

The directional derivative is approximately zero at $\theta = 236^o$ (where it takes value -0.00533) and $\theta = 56^o$ (where it takes value -0.00533).

The yellow half-line is tangent to blue level curve through the point and perpendicular to the purple half-line.

Problem 6

a) The direction is $\hat{\mathbf{u}} = \dfrac{\langle 8, 2, -1\rangle}{\sqrt{69}}$.

In fact the direction of fastest decrease is $\hat{\mathbf{u}} = \dfrac{\nabla g(2,1,10)}{|\nabla g(2,1,10)|}$ and $\nabla g(x,y,z) = \langle -4x, -2y, 1\rangle$, so that $\nabla g(2,1,10) = \langle -8, -2, 1\rangle$.

b) Using linear approximation, we find that the point is

$$P = \left(2 + \frac{8}{69}, 1 + \frac{2}{69}, 10 - \frac{1}{69}\right) = \left(\frac{146}{69}, \frac{71}{69}, \frac{689}{69}\right).$$

The exact value of g at p is $g(P) = -\dfrac{44}{1587} \approx -0.028$.

The value of g at $P_0 = (2,1,10)$ is $g(P_0) = 10 - 2 \cdot 2^2 - 1^2 = 1$.
If we move from P_0 in direction $-\nabla g(P_0)$, we end up at the point $P = P_0 - s\nabla g(P_0)$ with $s \geq 0$.
Hence the problem is: find $s \geq 0$ such that $g\left(P_0 - s\nabla g(P_0)\right) = 0$.
Using linear approximation, we have

$$\begin{aligned} g\left(P_0 - s\nabla g(P_0)\right) &\approx g(P_0) + g_x(P_0)\Delta x + g_y(P_0)\Delta y + g_z(P_0)\Delta z = \\ &= g(P_0) + g_x(P_0)[-sg_x(P_0)] + g_y(P_0)[-sg_y(P_0)] + g_z(P_0)[-sg_z(P_0)] = \\ &= g(P_0) + \nabla g(P_0) \cdot (-s\nabla g(P_0)) = g(P_0) - s|\nabla g(P_0)|^2. \end{aligned}$$

Hence we obtain $s = \dfrac{g(P_0)}{|\nabla g(P_0)|^2} = \dfrac{1}{69}$ and so $P = P_0 - s\nabla g(P_0) = (2,1,10) - \dfrac{1}{69}\langle -8, -2, 1\rangle = \left(2 + \dfrac{8}{69}, 1 + \dfrac{2}{69}, 10 - \dfrac{1}{69}\right)$.
Using a calculator we get the exact value $g(P) = -\dfrac{44}{1587} \approx -0.028$.