These are the solutions to Exam 3 of **18.02**, Spring 2006.

Notice that these solutions contain some explanations

in square parentheses [...] that were not required.

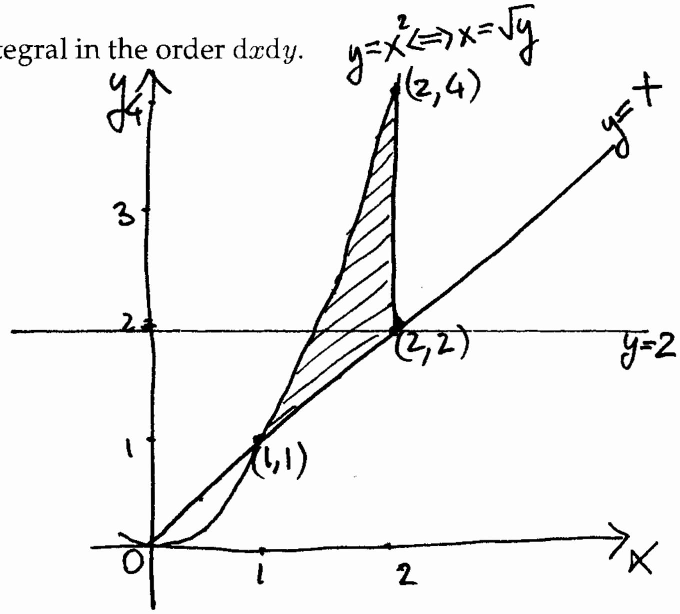
Problem 1

a) *(10)* Evaluate the integral $\int_1^2 \int_x^{x^2} 12x \, dy dx$.

$$\int_1^2\int_x^{x^2} 12x\,dy\,dx = \int_1^2 \Big[12xy\Big]_{y=x}^{y=x^2} dx = \int_1^2 (12x^3 - 12x^2)\,dx =$$

$$= \Big[3x^4 - 4x^3\Big]_1^2 = 48-32-3+4 = \boxed{17.}$$

b) *(15)* Sketch the region of integration, and express the integral in the order $dxdy$. Use two parts; do not evaluate.



Bottom:

$1 \le y \le 2$

$\sqrt{y} \le x \le y$

Top:

$2 \le y \le 4$

$\sqrt{y} \le x \le 2$

$$\boxed{\int_1^2\int_{\sqrt{y}}^{y} 12x\,dx\,dy + \int_2^4\int_{\sqrt{y}}^{2} 12x\,dx\,dy}$$

Problem 2

(15) Find the polar moment of inertia I_0 for the half-disk $x^2 + y^2 < a^2$, $x > 0$, with density $\delta(x, y) = x^2$.

$$I_0 = \int_{-\pi/2}^{\pi/2} \int_0^a x^2 \cdot r^2 \cdot r\,dr\,d\theta =$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^a r^5 \cdot \cos^2\theta \; dr\,d\theta =$$

$$= \int_{-\pi/2}^{\pi/2} \left[\frac{r^6}{6}\right]_{r=0}^{r=a} \cos^2\theta \, d\theta =$$

$$= 2 \int_0^{\pi/2} \frac{a^6}{6} \cos^2\theta \, d\theta =$$

$$= 2 \cdot \frac{a^6}{6} \cdot \frac{\pi}{4} = \boxed{\frac{\pi}{12} a^6}.$$

Problem 3

Let $\vec{\mathbf{F}} = axy\,\hat{\imath} + (e^y + 2x^2)\,\hat{\jmath}$.

a) *(5)* Find a so that $\vec{\mathbf{F}}$ is conservative.

$M = axy;\quad N = e^y + 2x^2$

$N_x = 4x;\quad M_y = ax$

$\vec{F}$ conservative $\Leftrightarrow N_x = M_y \Leftrightarrow \boxed{a = 4}$.

b) *(5)* For the value of a you found in part (a), find a potential function for $\vec{\mathbf{F}}$.

$\vec{F} = \nabla f = 4xy\,\hat{\imath} + (e^y + 2x^2)\,\hat{\jmath}$.

$f_x(x,y) = 4xy \Rightarrow f(x,y) = 2x^2y + g(y)$.

$f_y(x,y) = e^y + 2x^2 \Rightarrow 2x^2 + g'(y) = e^y + 2x^2$

$\Rightarrow g'(y) = e^y \Rightarrow g(y) = e^y + C$.

Hence, $\boxed{f(x,y) = 2x^2y + e^y + C}$ is a potential for every constant C.

[Full credit without C.]

c) *(5)* For the same value of a as in parts (a) and (b), find the work done by $\vec{\mathbf{F}}$ along the path $x = t,\ y = \cos t,\ 0 \le t \le \pi$.

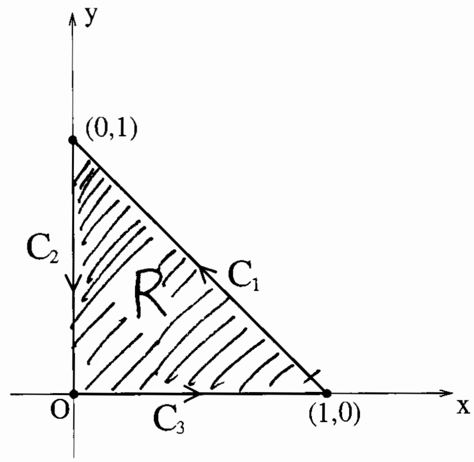
$$\int_{\text{path}} \vec{F}\cdot d\vec{r} = \int_{\text{path}} (\nabla f)\cdot d\vec{r} = f(\text{end point}) - f(\text{starting point}) =$$

$$= f(\pi, -1) - f(0, 1) =$$

$$= 2\pi^2(-1) + \frac{1}{e} - e = \boxed{\frac{1}{e} - e - 2\pi^2}$$

Problem 4

Suppose that $\vec{\mathbf{F}} = (2xy + y)\,\hat{\imath} + x^2\,\hat{\jmath}$ and $C = C_1 + C_2 + C_3$ is the loop around the triangle as pictured.



a) *(10)* Compute the line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ using Green's theorem.

$M = 2xy + y$; $N = x^2$

$M_y = 2x + 1$; $N_x = 2x$

$N_x - M_y = -1$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_R (\text{curl } \vec{F})\, dA = \iint_R (-1) \cdot dA =$$

$$= -\text{Area}(R) = \boxed{-\frac{1}{2}}$$

b) *(15)* Compute the same line integral directly from the definition without using part (a) or Green's theorem.

C_3: $\hat{T} = \hat{\imath}$; $\vec{F}\big|_{C_3} = x^2\hat{\jmath} \Rightarrow \underset{\text{along } C_3}{\hat{T} \cdot \vec{F} = 0} \Rightarrow \int_{C_3} \vec{F} \cdot d\vec{r} = 0.$

C_2: $\hat{T} = -\hat{\jmath}$; $\vec{F}\big|_{C_2} = y\,\hat{\imath} \Rightarrow \underset{\text{along } C_2}{\hat{T} \cdot \vec{F} = 0} \Rightarrow \int_{C_2} \vec{F} \cdot d\vec{r} = 0.$

C_1: $y = 1 - x$, $0 \le x \le 1$ (reversed orientation!)

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_1} (2xy + y)\,dx + x^2\,dy =$$

$$= \int_1^0 [2x(1-x) + (1-x)]\,dx + x^2(-dx) =$$

$$= -\int_0^1 (-3x^2 + x + 1)\,dx = \left[x^3 - \frac{x^2}{2} - x\right]_0^1 = -\frac{1}{2}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} + \int_{C_3} \vec{F} \cdot d\vec{r} = -\frac{1}{2}.$$

Problem 5

a) *(10)* Compute the Jacobian factor $dudv = \left|\dfrac{\partial(u,v)}{\partial(x,y)}\right| dxdy$

for the change of variable $u = x^2/y$, $v = xy$.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} 2x/y & -\frac{x^2}{y^2} \\ y & x \end{pmatrix}$$

$$\left|\det \frac{\partial(u,v)}{\partial(x,y)}\right| = \left|\frac{2x}{y}\cdot x + \frac{x^2}{y^2}\cdot y\right| = \left|\frac{3x^2}{y}\right| = \boxed{3\,|u|.}$$

b) *(10)* Use this change of variable to find the area of the region in the xy-plane given by $1 \leq x^2/y \leq 2$, $0 \leq xy \leq 1$.

$$\text{Area (Region)} = \iint_{\text{Region}} dx\,dy = \iint_{\text{Region}} \frac{1}{3|u|}\, dv\, du$$

Region: $1 \leq u \leq 2$
$0 \leq v \leq 1$

$$\text{Area} = \int_1^2 \int_0^1 \frac{1}{3u}\, dv\, du = \int_1^2 \frac{1}{3u}\,\Big[v\Big]_{v=0}^{v=1} du =$$

$$= \int_1^2 \frac{du}{3u} = \left[\frac{1}{3}\ln u\right]_1^2 = \boxed{\frac{1}{3}\ln 2.}$$