# 18.02 Problem Set 9 - Solutions of Part B

**Problem 1**

a) If $\overrightarrow{\mathbf{F}} = x\hat{\boldsymbol{\jmath}}$, then $\operatorname{curl}\overrightarrow{\mathbf{F}} = 1$. Hence $\text{Area}(R) = \iint_R 1 \text{ dA} = \iint_R \operatorname{curl}\overrightarrow{\mathbf{F}} \text{ dA} =$
$= \oint_C \overrightarrow{\mathbf{F}} \cdot \text{d}\vec{\mathbf{r}} = \oint_C x \text{ d}y$, where the third equal sign holds because of Green's theorem.
Similarly, if $\overrightarrow{\mathbf{F}} = -y\hat{\boldsymbol{\imath}}$, then $\operatorname{curl}\overrightarrow{\mathbf{F}} = 1$ and $\text{Area}(R) = \iint_R 1 \text{ dA} = \iint_R \operatorname{curl}\overrightarrow{\mathbf{F}} \text{ dA} =$
$= \oint_C \overrightarrow{\mathbf{F}} \cdot \text{d}\vec{\mathbf{r}} = \oint_C -y \text{ d}x$.
[Actually, one could use any $\overrightarrow{\mathbf{F}}$ well-defined and differentiable over $R$ such that $\operatorname{curl}\overrightarrow{\mathbf{F}} = 1$.]

b) The area is $3\pi a^2$.

Call $C_1$ the arc of cycloid and $C_2$ the segment from $(0,0)$ to $(2\pi a, 0)$, so that the boundary of $R$ (when counterclockwise oriented) is $-C_1 + C_2$.
[$-C_1$ means: $C_1$ with reversed orientation.]
Along $C_1$, $\text{d}x = a(1-\cos t)\text{d}t$. Along $C_2$, $y = 0$.
Using (a) we obtain

$$\text{Area}(R) = -\int_{C_1} -y \text{ d}x + \int_{C_2} -y \text{ d}x = \int_0^{2\pi} a(1-\cos t)a(1-\cos t)\text{d}t =$$

$$= \int_0^{2\pi} a^2(1 - 2\cos t + \cos^2 t)\text{d}t = a^2 \left[t - 2\sin t + \frac{t}{2} + \frac{\sin 2t}{4}\right]_0^{2\pi} = 3\pi a^2.$$

**Problem 2**

a) For $C$ equal to the circle of radius 2 centered at the origin (i.e. with equation $x^2 + y^2 = 4$).

Call $R$ the region of the plane enclosed by $C$.
If we define $\overrightarrow{\mathbf{F}} = (x^2y + y^3 - y)\hat{\boldsymbol{\imath}} + (3x + 2y^2x + e^y)\hat{\boldsymbol{\jmath}}$, then
$\operatorname{curl}\overrightarrow{\mathbf{F}} = (3 + 2y^2) - (x^2 + 3y^2 - 1) = 4 - x^2 - y^2$.

Using Green's theorem

$$\oint_C (x^2y + y^3 - y)\ \mathrm{d}x + (3x + 2y^2x + e^y)\ \mathrm{d}y = \iint_R (4 - x^2 - y^2)\mathrm{dA}$$

and the right hand side achieves its maximum value if $R$ is exactly the region of plane on which $4 - x^2 - y^2 \geq 0$. This happens if and only if $C$ has equation $x^2 + y^2 = 4$.

b) The maximum value is $8\pi$.

$$\iint_R (4 - x^2 - y^2)\ \mathrm{dA} = \int_0^{2\pi} \int_0^2 (4 - r^2)r\ \mathrm{d}r\mathrm{d}\theta = \int_0^{2\pi} \mathrm{d}\theta\ \int_0^2 (4r - r^3)\ \mathrm{d}r =$$
$$= 2\pi \left[2r^2 - \frac{r^4}{4}\right]_0^2 = 2\pi(8 - 4) = 8\pi.$$

**Problem 3**

a) Call $R_1$ the region enclosed by $C_1$. From *Problem Set 8 - Exercise 1(a)* we know (by direct computation) that $\oint_{C_1} \overrightarrow{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = 0$.

Now, $\mathrm{curl}\,\overrightarrow{\mathbf{F}} = 2xy - 2y$ and $\iint_{R_1} (2xy - 2y)\ \mathrm{dA} = 0$ because the reflection with respect to the $x$-axis $(x, y) \mapsto (x, -y)$ preserves $R_1$ and dA but switches sign to the integrand $(2xy - 2y)$.

b) Call $R_2$ the region enclosed by $C_2$. From *Problem Set 8 - Exercise 1(b)* we know (by direct computation) that $\oint_{C_2} \overrightarrow{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \frac{a^4}{12} - \frac{a^3}{3}$.

On the other hand, $\iint_{R_2} \mathrm{curl}\,\overrightarrow{\mathbf{F}}\ \mathrm{dA} = \int_0^a \int_0^{a-y} (2xy - 2y)\ \mathrm{d}x\,\mathrm{d}y =$

$$= \int_0^a [x^2y - 2xy]_{x=0}^{x=a-y}\ \mathrm{d}y = \int_0^a [(a - y)^2y - 2(a - y)y]\ \mathrm{d}y =$$
$$= \int_0^a (a^2y - 2ay^2 + y^3 - 2ay + 2y^2)\ \mathrm{d}y = \left[\frac{a^2 - 2a}{2}y^2 + \frac{2 - 2a}{3}y^3 + \frac{1}{4}y^4\right]_0^a =$$
$$= \frac{a^4}{2} - a^3 + \frac{2a^3 - 2a^4}{3} + \frac{a^4}{4} = \frac{a^4}{12} - \frac{a^3}{3}.$$