Solutions

NAME

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code

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Recitation:	☐ Rec.#1	☐ Rec.#7
	☐ Rec.#2	☐ Rec.#8
	☐ Rec.#3	☐ Rec.#9
	☐ Rec.#4	☐ Rec.#10
	☐ Rec.#6	☐ Rec.#11

18.02 Exam 1 Thursday, Feb 23rd, 2006

Directions: Do all the work on these pages; use reverse side if needed. Answers without accompanying reasoning may only receive partial credit. No books, notes, or calculators. Please stop when asked to and don't talk until your paper is handed in.

GRADING	
1.	/ 15
2.	/ 20
3.	/ 25
4.	/ 15
5.	/ 25
TOTAL	/ 100

Problem 1.

The plane $x - 2y + 2z = 2$ meets the x-axis at P, the y-axis at Q and the z-axis at R.

a) (7) Find the vectors $\overrightarrow{QP}$ and $\overrightarrow{QR}$ in terms of $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$.

$$P = (2,0,0) \quad , \quad Q = (0,-1,0), \quad R = (0,0,1)$$

$$\overrightarrow{QP} = P - Q = \langle 2, 1, 0 \rangle = 2\hat{\imath} + \hat{\jmath}$$

$$\overrightarrow{QR} = R - Q = \langle 0, 1, 1 \rangle = \hat{\jmath} + \hat{k}.$$

b) (8) Find the cosine of the angle PQR.

$$\cos\theta = \frac{\overrightarrow{QP} \cdot \overrightarrow{QR}}{|\overrightarrow{QP}| \cdot |\overrightarrow{QR}|} = \frac{\langle 2,1,0 \rangle \cdot \langle 0,1,1 \rangle}{\sqrt{4+1} \cdot \sqrt{1+1}} =$$

$$= \frac{1}{\sqrt{10}}.$$

Problem 2.

a) (10) Find the area of the triangle with vertices $P_1 = (1, -1, 0)$, $P_2 = (2, 1, 0)$, and $P_3 = (-2, 2, 2)$.

$\overrightarrow{P_1P_2} = P_2 - P_1 = \langle 1, 2, 0 \rangle$, $\quad \overrightarrow{P_1P_3} = P_3 - P_1 = \langle -3, +3, 2 \rangle$

$$\text{Area} = \frac{1}{2} \left| \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} \right| = \frac{1}{2} \left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 0 \\ -3 & 3 & 2 \end{vmatrix} \right\| = \frac{1}{2} \left| 4\hat{i} - 2\hat{j} + 9\hat{k} \right| =$$

$$= \frac{1}{2}\sqrt{16 + 4 + 81} = \frac{1}{2}\sqrt{101}.$$

b) (5) Find the equation for the plane through the points in the form $ax + by + cz = d$.

$$\text{Normal} \quad \vec{N} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 0 \\ -3 & 3 & 2 \end{vmatrix} = 4\hat{i} - 2\hat{j} + 9\hat{k}$$

Eqn. $4x - 2y + 9z = d$.

P_1 on plane $\Rightarrow 4 \cdot (1) - 2(-1) + 9 \cdot 0 = 6 = d$

$\Rightarrow$ Eqn. $\boxed{4x - 2y + 9z = 6}$

c) (5) Does the plane meet the segment from $Q_1 = (0, -1, 0)$ to $Q_2 = (2, 0, 0)$?
(Say yes or no and why.)

Yes.

Reason: the plane splits the space into two parts: $4x - 2y + 9z < 6$ and $4x - 2y + 9z > 6$.
Q_1 belongs to the first one (because $4 \cdot 0 - 2(-1) + 9 \cdot 0 = 2 < 6$) and Q_2 belongs to the second one (because $4 \cdot 2 - 2 \cdot 0 + 9 \cdot 0 = 8 > 6$).
Therefore the segment meets the plane. ∎

Problem 3.

a) (10) Find the entries a and b of the inverse matrix A^{-1}.

$$A = \begin{pmatrix} 0 & 3 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & 0 \end{pmatrix}; \qquad A^{-1} = \begin{pmatrix} \cdot & \cdot & \cdot \\ a & b & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

$$\det A = -3\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} + 1\cdot\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = -3(-2)+1\cdot 3 = 6+3 = 9$$

$$a = \frac{1}{\det A}(-1)\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = -\frac{1}{9}\cdot(-2) = \frac{2}{9}$$

$$b = \frac{1}{\det A}\begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} = \frac{1}{9}(-2) = -\frac{2}{9}$$

b) (8) For the system below, say which of x, y or z you can find using only the entries a and b from A^{-1}, and find it.

$$3y + z = 1$$
$$x - y + z = 3$$
$$2x + y = 0$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} * & * & * \\ a & b & * \\ * & * & * \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} * \\ a+3b \\ * \end{pmatrix},$$ So we can determine y.

$$y = a+3b = \frac{2}{9} + 3\left(-\frac{2}{9}\right) = -\frac{4}{9}.$$

c) (7) For which values of c does the system below have exactly one solution?

$$3y + cz = 1$$
$$x - y + z = 3$$
$$2x + y = 0$$

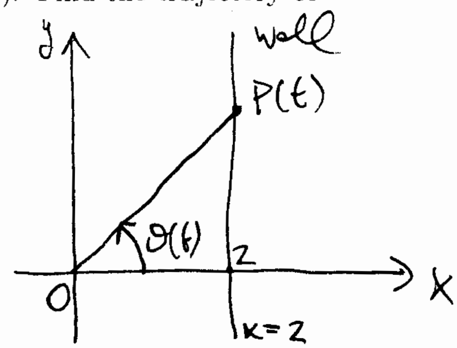
$$A = \begin{pmatrix} 0 & 3 & c \\ 1 & -1 & 1 \\ 2 & 1 & 0 \end{pmatrix}.$$ Exactly one solution when A^{-1} exists, that is when $\det A \neq 0$.

$$\det A = -3\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} + c\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 6+3c.$$

<u>Answer</u>: $\boxed{c \neq -2}$.

Problem 4.

a) (7) A searchlight at the origin points in the direction $\hat{\imath}$ at time $t = 0$ minutes and rotates counterclockwise in the plane at one rpm (revolution per minute). Find the trajectory of the path of the light on a wall $x = 2$.



$\vec{r}(t) = \vec{OP}(t) = \langle x(t), y(t) \rangle$.

Clearly $x(t) = 2$.

$\theta(t) = 2\pi t$, so

$y(t) = 2\tan(\theta(t)) = 2\tan(2\pi t)$.

$$\boxed{\vec{r}(t) = \langle 2, 2\tan(2\pi t) \rangle.}$$

b) (8) An extendable robot arm rotates at 5 rpm counterclockwise around the origin O in the plane. At time $t = 0$ minutes the arm is the segment OP of length 1 along the positive x-axis, and the arm lengthens at unit speed. Find the parametric equations for the end $P = (x, y)$ of the arm.

$\vec{r}(t) = \vec{OP}(t) = \langle x(t), y(t) \rangle$.

$\hat{r}(t) = \text{direction of } \vec{r}(t) = \langle \cos(10\pi t), \sin(10\pi t) \rangle$.

$|\vec{r}(t)| = 1 + t$, because $\frac{d}{dt}|\vec{r}(t)| = 1$

and $|\vec{r}(0)| = 1$.

Hence $\vec{r}(t) = \langle (1+t)\cos(10\pi t), (1+t)\sin(10\pi t) \rangle$,

that is

$$\boxed{\begin{aligned} x(t) &= (1+t)\cos(10\pi t) \\ y(t) &= (1+t)\sin(10\pi t) \end{aligned}}$$

Problem 5.

Let $\vec{r} = \cos(t^3)\hat{j} + \sin(t^3)\hat{k}$

a) (5) Compute the velocity $\vec{v}$.

$$\vec{v}(t) = \frac{d}{dt}\left(\cos(t^3)\hat{j} + \sin(t^3)\hat{k}\right) = -3t^2 \cdot \sin(t^3)\hat{j} + 3t^2 \cdot \cos(t^3)\hat{k}$$

(or, which is the same, $\langle 0, -3t^2 \cdot \sin(t^3), 3t^2 \cdot \cos(t^3)\rangle$.

b) (5) Compute $\vec{r} \cdot \vec{v}$.

$$\vec{r} \cdot \vec{v} = \langle 0, \cos(t^3), \sin(t^3)\rangle \cdot \langle 0, -3t^2\sin(t^3), 3t^2\cos(t^3)\rangle =$$
$$= -3t^2\cos(t^3)\sin(t^3) + 3t^2\sin(t^3)\cos(t^3) = 0.$$

(Shortcut: $\vec{r} \cdot \vec{v} = \frac{1}{2}\frac{d}{dt}(\vec{r} \cdot \vec{r}) = \frac{1}{2}\frac{d}{dt}(1) = 0.$)

c) (10) Compute $\vec{r} \times \vec{v}$.

$$\vec{r} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \cos(t^3) & \sin(t^3) \\ 0 & -3t^2\sin(t^3) & 3t^2\cos(t^3) \end{vmatrix} = \hat{i}\begin{vmatrix} \cos t^3 & \sin t^3 \\ -3t^2\sin t^3 & 3t^2\cos t^3 \end{vmatrix} - \hat{j} \cdot (0) + \hat{k} \cdot (0) =$$
$$= 3t^2 \cdot \hat{i}.$$

d) (5) Compute $\vec{r} \times \vec{a}$.

($\vec{a}$ is acceleration. Hint: It's faster to use (c).)

$$\frac{d}{dt}(\vec{r} \times \vec{v}) = \frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt} = \vec{v} \times \vec{v} + \vec{r} \times \vec{a}.$$

As $\vec{v} \times \vec{v} = 0$, $\vec{r} \times \vec{a} = \frac{d}{dt}(\vec{r} \times \vec{v}) = \frac{d}{dt}(3t^2\hat{i}) = 6t \cdot \hat{i}$.

(Direct computation with $\vec{a} = \frac{d\vec{v}}{dt}$ is much more complicated.)