These are the solutions to Exam 4 of **18.02**, Spring 2006.

Notice that these solutions contain some explanations
in parentheses that were not required.

Problem 1

(20) Find the mass of the solid cylinder $0 \le x^2+y^2 \le a^2$, $0 \le z \le b$, with density $\delta(x,y,z) = x^2 z$.

$$M = \iiint_{\text{cylinder}} \delta \cdot dV$$

Cylindrical coordinates: $\begin{cases} 0 \le r \le a \\ 0 \le z \le b \\ 0 \le \theta \le 2\pi \end{cases}$

$$\delta = x^2 z = (r\cos\theta)^2 z = r^2 z \cdot \cos^2\theta$$
$$dV = r\,dr\,d\theta\,dz$$

$$M = \int_0^b \int_0^{2\pi} \int_0^a r^2 z \cos^2\theta \cdot r\,dr\,d\theta\,dz =$$

$$= \int_0^b z\,dz \cdot \int_0^{2\pi} \cos^2\theta\,d\theta \cdot \int_0^a r^3\,dr =$$

$$= \left[\frac{z^2}{2}\right]_0^b \cdot \left(4 \cdot \frac{\pi}{4}\right) \cdot \left[\frac{r^4}{4}\right]_0^a =$$

$$= \frac{b^2}{2} \cdot \pi \cdot \frac{a^4}{4} = \frac{\pi}{8} a^4 b^2.$$

$\left[\int_0^{2\pi} \cos^2\theta\,d\theta = 4 \cdot \int_0^{\pi/2} \cos^2\theta\,d\theta = 4 \cdot \frac{\pi}{4} \text{ from the table.} \right]$

Problem 2

(15) Express the average value of z^{10} on the surface of the upper hemisphere $x^2 + y^2 + z^2 = 1$, $z > 0$, as an integral in spherical coordinates. **(Do not evaluate.)**

$$\text{Average of } (z^{10}) = \frac{1}{\text{Area}} \iint_{\text{surface}} z^{10} \cdot dS$$

$$\text{Upper hemisphere} = \begin{cases} \rho = 1 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \varphi \leq \pi/2 \end{cases}$$

$$\text{Area} = \frac{1}{2} 4\pi\rho^2 = 2\pi.$$

$z = \rho \cdot \cos\varphi = \cos\varphi$ on the hemisphere

$dS = \rho^2 \sin\varphi \, d\varphi \, d\theta$ ~~on the~~

$= \sin\varphi \, d\varphi \, d\theta$ on the hemisphere.

$$\text{Average of } (z^{10}) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\pi/2} \cos^{10}\varphi \cdot \sin\varphi \, d\varphi \, d\theta.$$

Problem 3

a) *(5)* Explain why $\boldsymbol{F} = \langle y, x + az, y + 1\rangle$ cannot be a gradient field unless $a = 1$.

$$\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$$

$$\vec{F} = \nabla f \Longrightarrow Q_z = R_y \ (\text{because } Q_z = f_{yz} = f_{zy} = R_y)$$

$$\Updownarrow$$

$$(x + az)_z = (y+1)_y \iff a = 1.$$

b) *(10)* Next, let $a = 1$. Then $\boldsymbol{F} = \langle y, x + z, y + 1\rangle = \nabla(xy + yz + z)$.

Find $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ for C given by $x = \cos^3 t$, $y = t$, $z = \sin^3 t$, $0 \le t \le \pi$.

Fundamental theorem of calculus for line integrals: if $\vec{F} = \nabla f$, then

$$\int_C \vec{F} \cdot d\vec{r} = f(\text{endpoint}) - f(\text{starting point}).$$

Endpoint $= (-1, \pi, 0)$ at $t = \pi$.

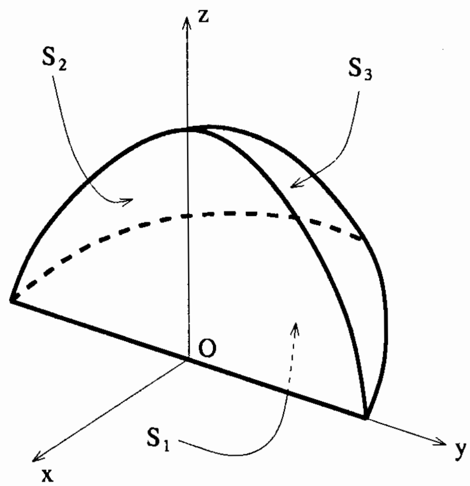
Starting point $= (1, 0, 0)$ at $t = 0$.

$$\int_C \vec{F} \cdot d\vec{r} = f(-1, \pi, 0) - f(1, 0, 0) = (-1 \cdot \pi) - 0 = -\pi.$$

Problem 4

Consider $\boldsymbol{F} = \hat{\imath}$ and D the solid quarter of a ball given by $x^2 + y^2 + z^2 < 1$, $x < 0$ and $z > 0$. Let $S = S_1 + S_2 + S_3$ denote the surface that encloses D, with S_1 the flat face in the xy-plane, S_2 the flat face in the yz-plane, and S_3 the curved face.

a) *(15)* State the divergence theorem, and use it to find the flux out of the curved face from the fluxes through the flat faces.



If S encloses D (and $\vec{F}$ is continuous and differentiable on S and D), then

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_D (\operatorname{div} \vec{F})\, dV.$$

In our case, $\operatorname{div} \vec{F} = 0$.

So $\iiint_D (\operatorname{div} \vec{F})\, dV = 0$.

S_1: $\hat{n} = \hat{k} \Rightarrow \vec{F} \cdot \hat{n} = 0 \Rightarrow \iint_{S_1} (\vec{F} \cdot \hat{n})\, dS = 0$.

S_2: $\hat{n} = \hat{\imath} \Rightarrow \vec{F} \cdot \hat{n} = 1 \Rightarrow \iint_{S_2} (\vec{F} \cdot \hat{n})\, dS = \text{Area}(S_2) = \pi/2$.

$$\iint_{S_3} (\vec{F} \cdot \hat{n})\, dS = \iiint_D (\operatorname{div} \vec{F})\, dV - \iint_{S_1} (\vec{F} \cdot \hat{n})\, dS - \iint_{S_2} (\vec{F} \cdot \hat{n})\, dS =$$

$$= 0 - 0 - \frac{\pi}{2} = -\frac{\pi}{2}.$$

b) *(10)* Find the integrand $f(x, y)$ in the integral formula for the flux you found indirectly in part (a), that is,

$$\text{flux of } \boldsymbol{F} \text{ out of } S_3 = \iint_{x^2+y^2<1,\ x<0} f(x, y)\, dx\, dy$$

Do not evaluate the integral, and do not calculate the limits of integration. (The region of integration is the projection (shadow) of S_3 in the xy-plane.)

$g(x, y, z) = x^2 + y^2 + z^2$

$$d\vec{S} = \frac{\nabla g}{g_z}\, dx\, dy = \frac{\langle 2x, 2y, 2z \rangle}{2z}\, dx\, dy = \langle \tfrac{x}{z}, \tfrac{y}{z}, 1 \rangle\, dx\, dy.$$

$$\vec{F} \cdot d\vec{S} = \langle 1, 0, 0 \rangle \cdot \langle \tfrac{x}{z}, \tfrac{y}{z}, 1 \rangle\, dx\, dy = \tfrac{x}{z}\, dx\, dy$$

$$\iint_{x^2+y^2<1,\ x<0} \frac{x}{z}\, dx\, dy = \iint_{x^2+y^2<1,\ x<0} \frac{x}{\sqrt{1-x^2-y^2}}\, dx\, dy \quad \left(\text{since } z = \sqrt{1-x^2-y^2} \text{ on } S_3\right)$$

Problem 5

(25) Consider the surface S which is the portion of the plane $2y + z = 0$ in the cylinder $x^2 + y^2 \leq 1$. Its boundary curve C is the ellipse given by $x^2 + y^2 = 1$, $z = -2y$.
State Stokes' theorem, and confirm it by direct computation for $\boldsymbol{F} = z\hat{\imath}$ on S.

S = surface with boundary C
If S and C are oriented compatibly (and $\vec{F}$ is continuous and differentiable), then

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}.$$

Evaluate LHS

C: $x = \cos\theta$
$y = \sin\theta \quad , 0 \leq \theta \leq 2\pi$
$z = -2\sin\theta$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C z\, dx =$$

$$= \int_0^{2\pi} (-2\sin\theta)(-\sin\theta\, d\theta) =$$

$$= 2\int_0^{2\pi} \sin^2\theta\, d\theta = 8\int_0^{\pi/2} \sin^2\theta\, d\theta =$$

$$= 8 \cdot \frac{\pi}{4} = 2\pi \text{ (from the table)}$$

Evaluate RHS

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ z & 0 & 0 \end{vmatrix} = \hat{\jmath}.$$

$\vec{N} = 2\hat{\jmath} + \hat{k}$ = normal to S.

$$d\vec{S} = \frac{\vec{N}}{\vec{N} \cdot \hat{k}}\, dx\, dy = \vec{N}\, dx\, dy.$$

$$(\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = (\hat{\jmath} \cdot \vec{N})\, dx\, dy = 2\, dx\, dy$$

Shadow(S) = region in the xy-plane where $x^2 + y^2 < 1$

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \iint_{\text{Shadow}(S)} 2 \cdot dx\, dy =$$

$$= 2 \cdot \text{Area}(\text{shadow of } S) = 2\pi.$$

Problem 6 – *Extra credit (10 points)*

(10) Let $\boldsymbol{F} = y\hat{\imath} + 2z\hat{\jmath}$. Suppose that $\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = 0$ for every curve in the plane $ax + by + cz = d$. What can be said about a, b, c, and d?

S = region in the plane $ax+by+cz=d$ enclosed by C.

Using Stokes $\iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,dS = \int_C \vec{F}\cdot d\vec{r} = 0$.

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y & 2z & 0 \end{vmatrix} = \langle -2, 0, -1\rangle .$$

$\hat{n}$ is parallel to $\langle a, b, c\rangle$, so we get

$$0 = \iint_S \hat{n}\cdot\langle -2,0,-1\rangle\, dS \iff \hat{n}\cdot\langle -2,0,-1\rangle = 0 \iff \langle a,b,c\rangle\cdot\langle -2,0,-1\rangle = 0$$

that is, $\boxed{-2a - c = 0}$.