18.02 Final Exam Solutions

Problem 1.

a) Line L has direction vector $\mathbf{v} = \langle -1, 2, -3 \rangle$ which lies in $\mathcal{P}$.

To get a point P_0 on L take $t = 0 \Rightarrow P_0 = (1, 1, 2)$.

$\Rightarrow \overrightarrow{\mathbf{P_0Q}} = \langle -1, 1, 2 \rangle - \langle 1, 1, 2 \rangle = \langle -2, 0, 0 \rangle$ also lies in $\mathcal{P}$.

$\Rightarrow$ A normal to $\mathcal{P}$ is

$$\mathbf{n} = \mathbf{v} \times \overrightarrow{\mathbf{P_0Q}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 3 \\ -2 & 0 & 0 \end{vmatrix} = \mathbf{i}(0) - \mathbf{j}(-6) + \mathbf{k}(4) = \langle 0, 6, 4 \rangle.$$

So, the equation of $\mathcal{P}$ is

$$0(x-1) + 6(y-1) + 4(z-2) = 0 \quad \text{or} \quad 6y = 4z = 14 \quad \text{or} \quad 3y + 2z = 7.$$

b) $\mathbf{n}_Q = \langle 2, 1, 1 \rangle \Rightarrow \widehat{\mathbf{n}} = \frac{1}{\sqrt{6}} \langle 2, 1, 1 \rangle$, $\quad \mathbf{v} = \langle -1, 2, -3 \rangle \Rightarrow \widehat{\mathbf{v}} = \frac{1}{\sqrt{14}} \langle -1, 2, -3 \rangle$

Component of $\widehat{\mathbf{n}}$ on $\widehat{\mathbf{v}}$ is

$$\widehat{\mathbf{n}} \cdot \widehat{\mathbf{v}} = \frac{1}{\sqrt{6 \cdot 14}}(2 + 2 - 3) = -\frac{3}{\sqrt{84}}$$

Problem 2.

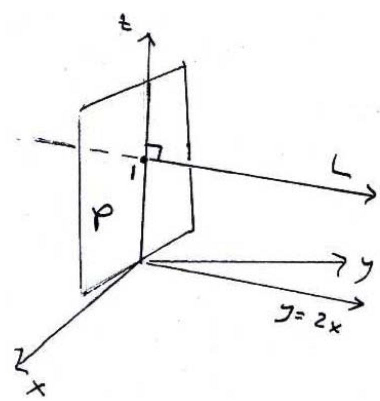
a) Direction vector for L: $\mathbf{v} = \langle 1, 2, 0 \rangle$.

$P_0 = (0, 0, 1) \Rightarrow$ equation for L:

$$\mathbf{r} = \langle x, y, z \rangle = \langle 0, 0, 1 \rangle + t \langle 1, 2, 0 \rangle$$

or

$$x = t, \quad y = 2t, \quad z = 1.$$



b) $\mathbf{n}$ = normal vector for $\mathcal{P} = \langle 1, 2, 0 \rangle$ since $L \perp \mathcal{P}$.

$P_0 = (0, 0, 1) \Rightarrow 1(x-0) + 2(y-0) + 0(z-1)$ or $x + 2y = 0$.

c) P on $L \Rightarrow P = (t, 2t, 1)$ for some $t \neq 0$ (part (a))

$P^* = (-t, -2t, 1)$ since then $\text{dist}(P_0, P) = \text{dist}(P_0, P^*) = |t|\sqrt{5}$.

Problem 3.

a) $$\begin{vmatrix} 1 & 0 & 3 \\ -2 & 1 & -1 \\ -1 & 1 & 2 \end{vmatrix} = 1 \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} - 0 \begin{vmatrix} -2 & -1 \\ -1 & 2 \end{vmatrix} + 3 \begin{vmatrix} -2 & 1 \\ -1 & 1 \end{vmatrix} = 3 - 3 = 0.$$

b) To get a non-zero solution take the cross-product of any two rows of A_2; for example

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 3 \\ -2 & 1 & -1 \end{vmatrix} = \langle -3, -5, 1 \rangle$$

This implies all solutions to $A\mathbf{x} = \mathbf{0}$ are $\mathbf{x} = t \begin{pmatrix} -3 \\ -5 \\ 1 \end{pmatrix}$.

c)

$$A_1^{-1} A_1 = I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} * & * & * \\ -3 & p & 5 \\ * & * & * \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ -2 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} * & * & * \\ -3-2p-5 & * & * \\ * & * & * \end{pmatrix} \Rightarrow -8-2p = 0 \Rightarrow p = -4.$$

Problem 4.

a) $\mathbf{r}'(t) = \langle -\sin(\mathrm{e}^t)\mathrm{e}^t, \cos(\mathrm{e}^t)\mathrm{e}^t, \mathrm{e}^t \rangle \Rightarrow |\mathbf{r}'(t)| = \mathrm{e}^t\sqrt{1+1} = \mathrm{e}^t\sqrt{2}$

$\Rightarrow \mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{2}} \langle -\sin(\mathrm{e}^t), \cos(\mathrm{e}^t), 1 \rangle.$

b)

$$\mathbf{T}'(t) = \frac{1}{\sqrt{2}} \langle -\cos(\mathrm{e}^t), -\sin(\mathrm{e}^t), 0 \rangle = -\frac{\mathrm{e}^t}{\sqrt{2}} \langle \cos(\mathrm{e}^t), \sin(\mathrm{e}^t), 0 \rangle.$$

Problem 5.

a) $F_x = \dfrac{\partial F}{\partial x} = \dfrac{xz}{(x^2+y)^{1/2}} \Rightarrow F_x(1,3,23) = 2/2 = 1.$

$F_y = \dfrac{\partial F}{\partial y} = \dfrac{z}{2(x^2+y)^{1/2}} + \dfrac{2}{z} \Rightarrow F_y(1,3,2) = 3/2.$

$F_z = \dfrac{\partial F}{\partial z} = (x^2+y)^{1/2} - \dfrac{2y}{z^2} \Rightarrow F_z(1,3,2) = \dfrac{1}{2}.$

$\mathbf{n} = \boldsymbol{\nabla} F(1,3,2) = \left\langle 1, \dfrac{3}{2}, \dfrac{1}{2} \right\rangle$, $P_0 = (1,3,2)$

$\Rightarrow$ tangent plane equation

$$1(x-1) + \frac{3}{2}(y-3) + \frac{1}{2}(z-2) \quad \text{or } 2x + 3y + z = 13.$$

b) At $P_0 = (1,3,2)$ we have $|F_y| = 3/2 > |F_x|, |F_z|$. So, a change in y produces the largest change in F.

$\Delta F = F_y \Delta y = \dfrac{3}{2}(0.1) = 0.15.$

c) $\left.\dfrac{df}{ds}\right|_{P_0,\mathbf{u}} = \widehat{\mathbf{u}} \cdot \boldsymbol{\nabla} F(P_0) = \pm\dfrac{1}{3}\langle -2, 2, 1 \rangle \cdot \langle 1, 3/2, 1/2 \rangle = \pm\dfrac{1}{3}(-2+3-1/2) = \pm\dfrac{1}{6}.$

$\Delta F \approx \left.\dfrac{dF}{ds}\right|_{P_0,\widehat{\mathbf{u}}} \Delta s \Rightarrow 0.1 = \dfrac{1}{6}\Delta s \Rightarrow \boxed{\Delta s = 0.6}$

Problem 6. a)

$$\left.\begin{array}{rcl} f_x &=& 1-2/(x^2y)=0 \\ f_y &=& 4-2/(xy^2)=0 \end{array}\right\} \Rightarrow \left.\begin{array}{c} x^2y=2 \\ xy^2=1/2 \end{array}\right\} \Rightarrow x=4y$$

$$\Rightarrow\ 4y^3=\frac{1}{2}\ \Rightarrow\ y^3=\frac{1}{8}\ \Rightarrow\ y=\frac{1}{2}\ \Rightarrow\ x=2.$$

There is one critical point at $(x,y)=(2,1/2)$.

b) $f_{xx}=4/(x^2y)$, $\quad f_{yy}=4/(xy^3)$, $\quad f_{xy}=f_{yx}=2/(x^2y^2)$
$A=f_{xx}(2,1/2)=1$, $\quad C=f_{yy}(2,1/2)=16$, $\quad B=f_{xy}(2,1/2)=2$
$\Rightarrow\ AC-B^2=12>0$, $A>0\ \Rightarrow\ f$ has a relative minimum at $(2,1/2)$.

Problem 7.
$f(x,y,z)=\text{dist}^2=(x-x_0)^2+(y-y_0)^2+(z-z_0)^2$
subject to $g(x,y,z)=Ax+By+Cz=D$.
$\boldsymbol{\nabla} f=2\langle x-x_0,y-y_0,z-z_0\rangle$, $\quad \boldsymbol{\nabla} g=\langle A,B,C\rangle$.

$$\boldsymbol{\nabla} f=\lambda\boldsymbol{\nabla} g,\text{ and } g=D\ \Rightarrow\ \begin{array}{l} 2(x-x_0)=\lambda A \\ 2(y-y_0)=\lambda B \\ 2(z-z_0)=\lambda C \\ Ax+By+Cz=D. \end{array}$$

Problem 8. a)

$$\frac{\partial F}{\partial\phi}=\frac{\partial F}{\partial x}\frac{\partial x}{\partial\phi}+\frac{\partial F}{\partial y}\frac{\partial y}{\partial\phi}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial\phi}$$

$$\frac{\partial x}{\partial\phi}=\rho\cos\phi\cos\theta\ \Rightarrow\ x_\phi(2,\pi/4,-\pi/4)=2\cos(\pi/4)\cos(-\pi/4)=1.$$

$$\frac{\partial y}{\partial\phi}=\rho\cos\phi\sin\theta\ \Rightarrow\ y_\phi(2,\pi/4,-\pi/4)=2\cos(\pi/4)\sin(-\pi/4)=-1.$$

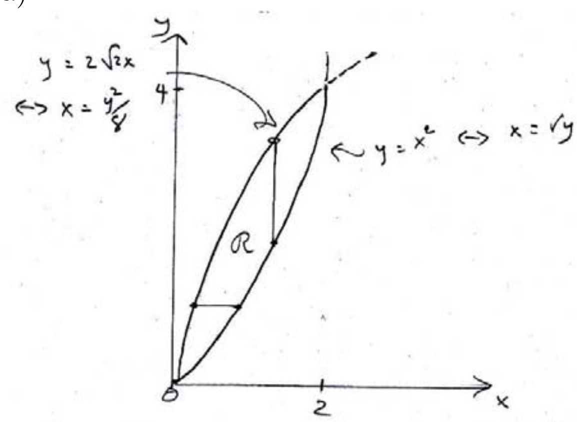
$$\frac{\partial z}{\partial\phi}=-\rho\sin\phi\ \Rightarrow\ z_\phi(2,\pi/4,-\pi/4)=-2\sin(\pi/4)=-\sqrt{2}.$$

b) **NOT Possible**
$\langle -y,x\rangle$ is *not* a gradient field. (Test: $\langle -y,x\rangle=\langle P,Q\rangle$: $\quad P_y=-1\neq Q_x=1$.)

Problem 9.

a)

$$R = \begin{cases} x^2 \le y \le 2\sqrt{x} \\ 0 \le x \le 2 \end{cases}$$



b)

$$R = \begin{cases} y^2/8 \le x \le \sqrt{y} \\ 0 \le y \le 4 \end{cases} \quad \Rightarrow \quad \int\int_R f\,dA = \int_0^4 \int_{y^2/8}^{\sqrt{y}} f(x,y)\,dx\,dy.$$

Problem 10.

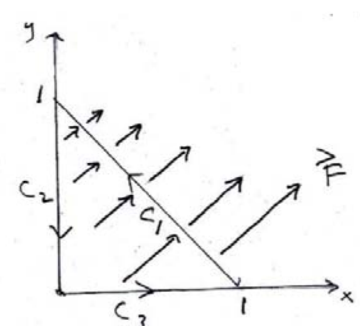
$$R_{u,v} : \begin{cases} 4 \le u \le 9 \\ 1 \le v \le 2 \end{cases}$$

Jacobian $J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} u^{-2/3}v^{-1/3}/3 & -u^{1/3}v^{-4/3}/3 \\ u^{-2/3}v^{2/3}/3 & 2u^{1/3}v^{-1/3}/3 \end{vmatrix} = \left(\frac{2}{9} + \frac{1}{9}\right) u^{-1/3}v^{-2/3} = \frac{1}{3}u^{-1/3}v^{-2/3}.$

$$\int_R f(x,y)\,dA = int_1^2 \int_4^9 f(u^{1/3}v^{-1/3}, u^{1/3}v^{2/3})(\frac{1}{3}u^{-1/3}v^{-2/3})\,du\,dv.$$

Problem 11.

a) Net flux out of R will be positive (more flow out than into R)



b)

$$\int_C \mathbf{F} \cdot \widehat{\mathbf{n}}\,ds = \int_C -N\,dx + M\,dy = \int_C -x\,dx + x\,dy$$

$C_1 : x = 1 - t,\ y = t \ \Rightarrow\ dx = -dt,\ dy = dt$

$$\Rightarrow \int_{C_1} \mathbf{F} \cdot \widehat{\mathbf{n}}\,ds = \int_0^1 -(1-t)(-1) + (1-t)(1)\,dt = -(1-t)^2\Big|_0^1 = 1.$$

$C_2 : x = 0 \;\Rightarrow\; \displaystyle\int_{C_2} \mathbf{F}\cdot\mathbf{n}\,ds = 0.$

$C_3 : y = 0, dy = 0 \;\Rightarrow\; \displaystyle\int_{C_3} \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^1 -x\,dx = -\frac{x^2}{2} = -\frac{1}{2}.$

Thus,

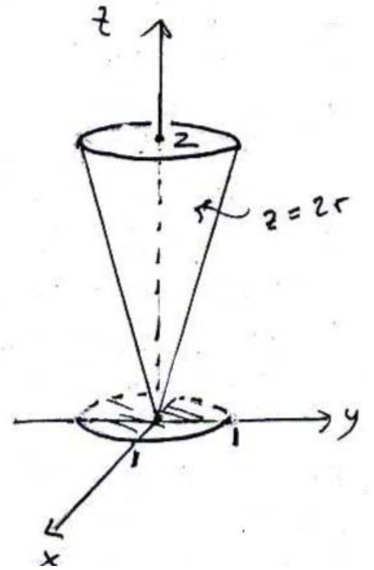
$$\oint_C \mathbf{F}\cdot\widehat{\mathbf{n}}\,ds = \int_{C_1+C_2+C_3} = 1 + 0 + (-1/2) = \frac{1}{2}.$$

c)

$$\operatorname{div}(\mathbf{F}) = M_x + N_y = 1 \;\Rightarrow\; \int\int_R \operatorname{div}(\mathbf{F})\,dA = \int\int_R dA = area(R) = \frac{1}{2}\cdot 1\cdot 1 = \frac{1}{2}$$

Problem 12.

a) Limits on G are $2r \le z \le 2$; $\;0 \le r \le 1$; $\;0 \le \theta \le 2\pi$.



In cylindrical coordinates $dV = dz\,r\,dr\,d\theta$.

Thus

$$M = \int\int\int_G z\,dV = \int_0^{2\pi}\int_0^1\int_{2r}^2 z\,dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 2(1-r^2)\,r\,dr\,d\theta = 4\pi\cdot\frac{1}{4} = \pi.$$

b)

$$\bar{z} = \frac{1}{M}\int\int\int_G z\cdot\delta\,dV = \frac{1}{\pi}\int_0^{2\pi}\int_0^1\int_{2r}^2 z^2\,dz\,r\,dr\,d\theta.$$

c) In spherical coordinates: $z = 2 \;\Rightarrow\; \rho\cos\phi = 2 \;\Rightarrow\; \rho = 2\sec\phi$.

Limits on G: $0 \le \rho \le 2\sec\phi$; $\;0 \le \phi \le \tan^{-1}(1/2)$; $\;0 \le \theta \le 2\pi$.

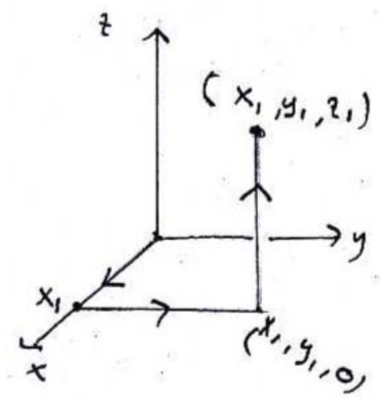
In spherical coordinates: $dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$ and $z = \rho\cos\phi$, so

$$\bar{z} = \frac{1}{M}\int\int\int_G z\cdot\delta\,dV = \frac{1}{\pi}\int_0^{2\pi}\int_0^{\tan^{-1}(1/2)}\int_0^{2\sec\phi}(\rho\cos\phi)^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

Problem 13.
a) We have $\mathbf{F} = \langle P, Q, R \rangle$, where $P = y + y^2 z, \quad Q = x - z + 2xyz, \quad R = -y + xy^2$.

$$\frac{\partial P}{\partial z} = y^2 = \frac{\partial R}{\partial x}; \quad \frac{\partial Q}{\partial z} = -1 + 2xy = \frac{\partial R}{\partial y}; \quad \frac{\partial P}{\partial y} = 1 + 2yz = \frac{\partial Q}{\partial x}.$$

b)



$$f(x_1, y_1, z_1) = \int_0^{x_1} P(x,0,0)\,dx + \int_0^{y_1} Q(x_1, y, 0)\,dy + \int_0^{z_1} Q(x_1, y_1, z)\,dz.$$

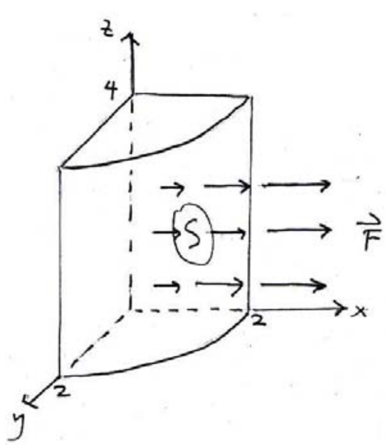
$P(x,0,0) = 0; \quad Q(x_1, y, 0) = x_1; \quad R(x_1, y_1, z) = -y_1 + x_1 y_1^2.$

$$f(x_1, y_1, z_1) = 0 + \int_0^{y_1} x_1\,dy + \int_0^{z_1} (-y_1 + x_1 y_1^2)\,dz$$

$\Rightarrow \; f(x_1, y_1, z_1) = x_1 y_1 - y_1 z_1 + x_1 y_1^2 z_1 \; \Rightarrow \; f(x,y,z) = xy - yz + xy^2 z + C.$

c) $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = f(1,-1,2) - f(2,2,1) = -10 + 3 = -7.$

Problem 14. a)



b) $\widehat{\mathbf{n}} = \frac{1}{2}\langle x, y, 0 \rangle, \quad \mathbf{F} = \langle x, 0, 0 \rangle.$

Thus, $\mathbf{F} \cdot \widehat{\mathbf{n}} = \dfrac{1}{2} x^2$ and in cylindrical coordinates $dS = 2dz\,d\theta$.

On the surface $x = 2\cos\theta$ and the limits of integration are $0 \le z \le 4$, and $0 \le \theta \le \pi/2$

$$\int\int_S \mathbf{F}\cdot\widehat{\mathbf{n}}\, dS = \frac{1}{2}\int\int_S x^2\, dS = \frac{1}{2}\int_0^{\pi/2}\int_0^4 (2\cos\theta)^2\, dz\, 2d\theta = 4\int_0^4 dz \int_0^{\pi/2}\cos^2(\theta)\, d\theta = 16\cdot\frac{\pi}{4}.$$

(We used the half angle formula $\cos^2\theta = \frac{1}{2}(1+\cos\theta)$.

c) $\mathrm{div}(\mathbf{F}) = \boldsymbol{\nabla}\cdot\mathbf{F} = 1 \;\Rightarrow\; \int\int\int_G \boldsymbol{\nabla}\cdot\mathbf{F}\, dV = \int\int\int_G 1\, dV = \mathrm{Vol}(G) = \frac{1}{4}\pi 2^2\cdot 4 = 4\pi.$

d) Flux of $\mathbf{F}$ across all four flat faces of G is zero.

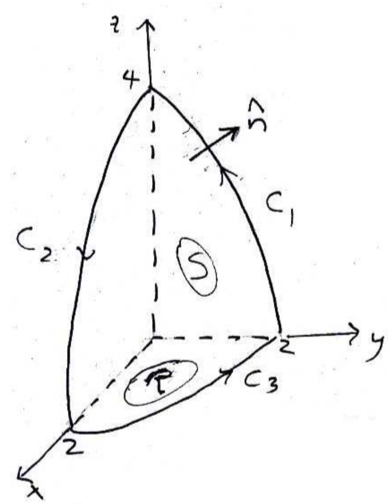
Check:

face on xz-plane: $\widehat{\mathbf{n}} = -\mathbf{j} \;\Rightarrow\; \mathbf{F}\cdot\widehat{\mathbf{n}} = 0$.

face on yz-plane: $\widehat{\mathbf{n}} = -\mathbf{i} \;\Rightarrow\; \mathbf{F}\cdot\widehat{\mathbf{n}} = -x = 0$ on $yz -$ plane.

faces on xy-plane and plane $z = 4$: $\widehat{\mathbf{n}} = -\mathbf{k}$ and $\mathbf{k}$ respectively, in either case $\mathbf{F}\cdot\widehat{\mathbf{n}}$.

Problem 15. a)



$$\boldsymbol{\nabla}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & -xy & 1 \end{vmatrix} = \mathbf{i}(x) - j(-y) + \mathbf{k}(-2z) = \langle x, y, -2z\rangle.$$

$\widehat{\mathbf{n}}\, dS = \langle -z_x, -z_y, 1\rangle\, dA = \langle 2x, 2y, 1\rangle\, dA.$

$(\boldsymbol{\nabla}\times\mathbf{F})\cdot\widehat{\mathbf{n}}\, dS = (2x^2+2y^2-2z)\, dA = (\text{subst. for } z\ ) = (2x^2+2y^2-2(4-x^2-y^2))\, dA = 4(x^2+y^2-2)\, dA.$

Limits of integration on R are $0 \le r \le 2$; $\quad 0 \le \theta \le \pi/2$.

$$\int\int_S(\boldsymbol{\nabla}\times\mathbf{F})\cdot\widehat{\mathbf{n}}\, dS = 4\int\int_R (x^2+y^2-2)\, dA = 4\int_0^{\pi/2}\int_0^2 (r^2-2)r\, dr\, d\theta = 4\cdot\frac{\pi}{2}\left(\frac{r^4}{4} - r^2\right|_0^2 = 2\pi(4-4) = 0.$$

b) $\mathbf{F} = \langle yz, -xz, 1\rangle \;\Rightarrow\; \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (yz)\, dx - (xz)\, dy + 1\, dz.$

C_1 is in the yz-plane: $x = 0$, $dx = 0$, $y = t$, $z = 4 - t^2$, $dz = (-2t)\,dt$ t goes from 2 to 0.

$$\Rightarrow \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} 1\,dz = \int_2^0 (-2t)\,dt = 4.$$

C_2 is in the xz-plane: $y = 0$, $dy = 0$, $x = t$, $z = 4 - t^2$, $dz = (-2t)\,dt$ t goes from 0 to 2.

$$\Rightarrow \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} 1\,dz = \int_0^2 (-2t)\,dt = -4.$$

C_3 is in the xy-plane: $z = 0$, $dz = 0$

$$\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = 0.$$

Thus,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1+C_2+C_2} \mathbf{F} \cdot d\mathbf{r} = 4 + (-4) = 0.$$