DAVID JORDAN: Hello, and welcome back to recitation. Today the problem I'd like to work with you is about computing partial derivatives and the total differential.

So we have a function z which is x squared plus y squared. So it depends on the two variables x and y. Now the variables x and y themselves depend on two auxiliary variables, u and v. So that's the setup that we have.

So in part a, we just want to compute the total differential dz in terms of dx and dy. So u and v aren't going to enter into the picture. And then in part b, we're going to compute the partial derivative partial z partial u in two different ways. First, we're going to compute it using the chain rule. And then we're going to compute it using total differentials. And so we'll substitute in some of the work that we had in a to solve that part.

So why don't you pause the video now and work on the problem. We'll check back and we'll do it together.

Hi, and welcome back. Let's get started.

So first, computing a is not so bad. So we just need to first remember, what does it mean to compute the total differential? So the total differential dz is just the partial derivative of z in the x-direction dx plus z in the y-direction dy. OK? So now, looking at our formula here for z, we have-- so the partial derivative of z in the x-direction is 2x, so this is 2x dx. And the partial derivative of z in the y is 2y, so we have 2y dy. OK, and that's all we have to do for a.

Now for b, we want to compute this partial derivative in two different ways. First, using the chain rule. So let's remember what the chain rule says.

So whenever I think about the chain rule, I like to draw this dependency graph. OK? And this is just a way for me to organize how the different variables depend on one another. So at the top, we have z. And z is a function of x and y, but x is itself a function of both u and v, and y is also a function of u and v. So z depends on x and y, and x and y each jointly depend on u and v. So it's a little bit complicated, the relationships here.

So now, what the chain rule says is that if we take a partial derivative-- partial z partial u-- we have to go through our dependency graph. Every way that we can get from z to u, we get a term in our summation for each one of those. So for instance, z goes to x goes to u. So that

means that we have partial z partial x, partial x partial u. And then we can also go z goes to y goes to u. And that will give us partial z partial y, partial y partial u.

And now these partials are ones that we can just compute from our formulas. So for instance, partial z partial x is 2x, which we computed. Now partial x partial u, we have to remember that x is defined as u squared minus v squared. And so partial x partial u, that's 2u. Partial z partial y, again, is this 2y that we computed. And partial y partial u is v. This v is just because u was u*v, and we take a partial in the u-direction. OK.

So altogether this is 4u*x plus 2v*y, and that's our partial derivative. So notice that, you know, x is a function of u and v. So if I really wanted to, I could substitute for x its formula for u and v, but that's not really necessary. You know, what's interesting about these problems is how the differentials depend on one another, and I'm perfectly happy with an answer that has mixed variables like this. That's fine.

So now, let's go over here and let's see if we can get the same answer by using total differentials. Now, I have to say that the chain rule that we used on the previous problem, it's the quickest way to do these sorts of things. I like to do total differentials if I have some time to actually explore the problem and get comfortable with it. I prefer to use total differentials because I think it's a little bit clearer. Somehow, this chain rule it's just, to me, it's just a prescription, it's not an explanation.

So why don't we compute some total differentials. So we already saw-- let me just repeat over here. We already saw that dz is 2x dx plus 2y dy. OK. Now, we want to use the fact that x is itself a function of u and v. So that's what we need to do now.

So that tells us that dx is 2u du minus 2v dv in the same way. And dy. So remember, y was u*v. So taking d of u*v, we get v du plus u dv. OK? So now, so what we've done is we've just listed out all of the total differentials.

And the nice thing about this is once you've done these computations, now it's just substitution. So what we really want to know is how does z depend on u and v. And so all we need to do is substitute in our formulas for dx here.

So this tells us that dz is-- OK, so we have 2x-- instead of dx, we just plug in here-- so we have 2u du minus 2v dv. So that was this term. And now we have plus 2y-- and now we just plug in this-- so v du plus u dv. You see? It's just substitution. So then now, we just expand everything

out. And so we get-- OK, so let's collect all the things involving du.

So if we collect all the things involving du, we have-- 2 times 2 times x times u-- 4x*u plus 2y*v. This whole quantity times du. And then if we collect the terms in dv, we have 2y*u. So that's coming from here, and then we have a minus 4x*v. OK? And now what that tells us is that-- so let's just remember that one definition of the partial derivative partial z partial u is this coefficient.

So if I go over here, if we write the total differential dz, we can write that as partial z partial u du plus partial z partial v dv. Right? Well, look. What we have here on these two sides is essentially the same expression. So that means if we want to compute partial z partial u, then that's just equal to this coefficient here.

So we get that partial z partial u is 4x*u plus 2-- that should be v. One of those is an x. Let's see. So where did this come from. Yeah, one of those is an x, sorry--

SPEAKER 1:   It's a y.

DAVID JORDAN: --is a y. 2v*y, OK. Now just as a sanity check, why don't we go back to the middle of the board, and we'll see that we got the same thing. So 4x*u plus 2v*y, that's what we concluded for partial z partial u. And then going back to the middle of the board, that's we found again.

So let's just go over the two different methods and compare them. So if I'm in a rush to do a computation-- maybe I'm taking an exam-- I definitely think it's the quickest to just compute, to figure out what the dependency of the variable is, and I use this dependency graph. And then I just trace all the paths from z to the independent variable u that I'm interested in. And then I multiply all the partial derivatives that correspond to each edge and I get an expression.

Now if I have more time, then I really prefer to use the method of total differentials that we did on the third board. I like it, because once you do some simple calculus, and then after that it's just, it's basic algebra. I find that I'm less likely to make a mistake doing that method. But as you saw, it involves computing a lot more derivatives that we didn't actually use in the final answer. For instance, when we computed total differentials, we got an expression for partial z partial v at the end of the day, even though we weren't asked to do that. So it's lengthier, but I think more conceptually straightforward.

So I think I'll leave it at that.