Uses of dot product

1. Find the angle between $\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and $2\mathbf{i}-\mathbf{j}+\mathbf{k}$.

<u>Answer:</u> We call the angle θ and use both ways of computing the dot product. Algebraically we have

$$(\mathbf{i}+\mathbf{j}+2\mathbf{k})\cdot(2\mathbf{i}-\mathbf{j}+\mathbf{k}) = 2-1+2 = 3.$$

Geometrically

$$(\mathbf{i}+\mathbf{j}+2\mathbf{k})\cdot(2\mathbf{i}-\mathbf{j}+\mathbf{k}) = |\mathbf{i}+\mathbf{j}+2\mathbf{k}|\cdot|2\mathbf{i}-\mathbf{j}+\mathbf{k}|\cos\theta = \sqrt{6}\,\sqrt{6}\cos\theta.$$

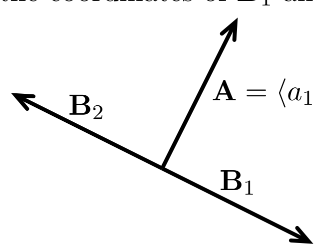
Combining these two we have

$$6\cos\theta = 3 \;\Rightarrow\; \cos\theta = \frac{3}{6} = \frac{1}{2} \;\Rightarrow\; \theta = \frac{\pi}{3}.$$

2. a) Are $\langle 1,3\rangle$ and $\langle -2,2\rangle$ orthogonal?

b) For what value of a are the vectors $\langle 1,a\rangle$ and $\langle 2,3\rangle$ at right angles?

c) In the figure the vectors $\mathbf{A}$ and $\mathbf{B}_1$ are orthogonal as are $\mathbf{A}$ and $\mathbf{B}_2$. If all the vectors are the same length what are the coordinates of $\mathbf{B}_1$ and $\mathbf{B}_2$?



<u>Answer:</u> a) Vectors are orthogonal if their dot product is 0. So, taking the dot product

$$\langle 1,3\rangle\cdot\langle -2,2\rangle = -2+6 = 4 \neq 0.$$

Thus the vectors are not orthogonal.

b) Setting the dot product to 0 and solving for a we get

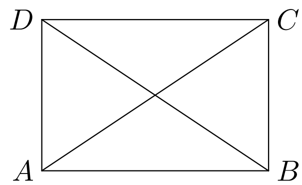
$$\langle 1,a\rangle\cdot\langle 2,3\rangle = 2+3a = 0 \;\Rightarrow\; a = -2/3.$$

c) $\mathbf{B}_1$ is $\mathbf{A}$ rotated 90° clockwise. We will show that $\mathbf{B}_1 = \langle a_2, -a_1\rangle$. It is easy to check that $|\langle a_2,-a_1\rangle| = |\mathbf{A}|$ and $\langle a_2,-a_1\rangle\cdot\mathbf{A} = 0$. The figure above shows that putting the negative sign on the a_1 means $\langle a_2,-a_1\rangle$ is turned clockwise from $\mathbf{A}$. Thus, $\langle a_2,-a_1\rangle = \mathbf{B}_1$.

$\mathbf{B}_2$ is $\mathbf{A}$ rotated 90° counterclockwise. Similarly to $\mathbf{B}_1$, we find $\mathbf{B}_2 = \langle -a_2, a_1\rangle$.

3. Using vectors and dot product show the diagonals of a parallelogram have equal lengths if and only if it's a rectangle

Answer:



We will make use of two properties of the dot product

1. $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$.

2. $\mathbf{v} \cdot \mathbf{w} = 0 \;\Leftrightarrow\; \mathbf{v} \perp \mathbf{w}$.

Referring to the figure, we will also need to use the fact that $ABCD$ is a parallelogram. That is, $\overrightarrow{\mathbf{AB}} = \overrightarrow{\mathbf{DC}}$.

We have $\overrightarrow{\mathbf{AC}} = \overrightarrow{\mathbf{AB}} + \overrightarrow{\mathbf{BC}}$ and $\overrightarrow{\mathbf{BD}} = \overrightarrow{\mathbf{BC}} + \overrightarrow{\mathbf{CD}} = \overrightarrow{\mathbf{BC}} - \overrightarrow{\mathbf{AB}}$.

Taking dot products:

$$|\overrightarrow{\mathbf{AC}}|^2 = \overrightarrow{\mathbf{AC}} \cdot \overrightarrow{\mathbf{AC}} = (\overrightarrow{\mathbf{AB}} + \overrightarrow{\mathbf{BC}}) \cdot (\overrightarrow{\mathbf{AB}} + \overrightarrow{\mathbf{BC}}) = |\overrightarrow{\mathbf{AB}}|^2 + 2\overrightarrow{\mathbf{AB}} \cdot \overrightarrow{\mathbf{BC}} + |\overrightarrow{\mathbf{BC}}|^2.$$

and

$$|\overrightarrow{\mathbf{BD}}|^2 = \overrightarrow{\mathbf{BD}} \cdot \overrightarrow{\mathbf{BD}} + (\overrightarrow{\mathbf{BC}} - \overrightarrow{\mathbf{AB}}) \cdot (\overrightarrow{\mathbf{BC}} - \overrightarrow{\mathbf{AB}}) = |\overrightarrow{\mathbf{BC}}|^2 - 2\overrightarrow{\mathbf{BC}} \cdot \overrightarrow{\mathbf{AB}} + |\overrightarrow{\mathbf{AB}}|^2$$

Comparing the two equations above we see

$$|\overrightarrow{\mathbf{AC}}|^2 = |\overrightarrow{\mathbf{BD}}|^2 \;\Leftrightarrow\; 4\overrightarrow{\mathbf{AB}} \cdot \overrightarrow{\mathbf{BC}} = 0.$$

This shows the diagonals have the same length if and only if $\overrightarrow{\mathbf{AB}} \perp \overrightarrow{\mathbf{BC}}$. That is, if and only if the sides of the parallelogram are orthogonal to each other. QED