Problems: Lagrange Multipliers

1. Find the maximum and minimum values of $f(x, y) = x^2 + x + 2y^2$ on the unit circle.

Answer: The objective function is $f(x, y)$. The constraint is $g(x, y) = x^2 + y^2 = 1$.

$$\begin{array}{llll} \text{Lagrange equations:} & f_x = \lambda g_x & \Leftrightarrow & 2x + 1 = \lambda 2x \\ & f_y = \lambda g_y & \Leftrightarrow & 4y = \lambda 2y \\ \text{Constraint:} & & & x^2 + y^2 = 1 \end{array}$$

The second equation shows $y = 0$ or $\lambda = 2$.

$\lambda = 2 \Rightarrow x = 1/2,\ y = \pm\sqrt{3}/2$.

$y = 0 \Rightarrow x = \pm 1$.

Thus, the critical points are $(1/2, \sqrt{3}/2)$, $(1/2, -\sqrt{3}/2)$, $(1, 0)$, and $(-1, 0)$.

$f(1/2, \pm\sqrt{3/2}) = 9/4$ (maximum).

$f(1, 0) = 2$ (neither min. nor max).

$f(-1, 0) = 0$ (minimum).

2. Find the minimum and maximum values of $f(x, y) = x^2 - xy + y^2$ on the quarter circle $x^2 + y^2 = 1$, $x, y \geq 0$.

Answer: The constraint function here is $g(x, y) = x^2 + y^2 = 1$. The maximum and minimum values of $f(x, y)$ will occur where $\boldsymbol{\nabla} f = \lambda \boldsymbol{\nabla} g$ or at endpoints of the quarter circle.

$$\boldsymbol{\nabla} f = \langle 2x - y, -x + 2y \rangle \quad \text{and} \quad \boldsymbol{\nabla} g = \langle 2x, 2y \rangle.$$

Setting $\boldsymbol{\nabla} f = \lambda \boldsymbol{\nabla} g$, we get $2x - y = \lambda \cdot 2x$ and $-x + 2y = \lambda \cdot 2y$.

Solving for λ and setting the results equal to each other gives us:

$$\begin{aligned} \frac{2x - y}{2x} &= \frac{-x + 2y}{2y} \\ 2xy - y^2 &= -x^2 + 2xy \\ x^2 &= y^2. \end{aligned}$$

Because we're constrained to $x^2 + y^2 = 1$ with x and y non-negative, we conclude that $x = y = \frac{1}{\sqrt{2}}$.

Thus, the extreme points of $f(x, y)$ will be at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(1, 0)$, or $(0, 1)$.

$$f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{1}{2} \quad \text{is the minimum value of } f \text{ on this quarter circle.}$$

$$f(1, 0) = f(0, 1) = 1 \quad \text{are the maximal values of } f \text{ on this quarter circle.}$$