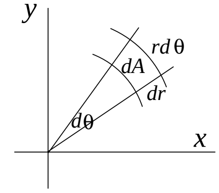
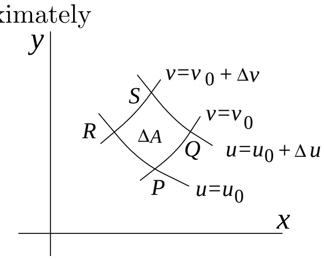
Changing Variables in Multiple Integrals

2. The area element.



In polar coordinates, we found the formula $dA = r\,dr\,d\theta$ for the area element by drawing the grid curves $r = r_0$ and $\theta = \theta_0$ for the r, θ-system, and determining (see the picture) the infinitesimal area of one of the little elements of the grid.

For general u, v-coordinates, we do the same thing. The grid curves (4) divide up the plane into small regions ΔA bounded by these contour curves. If the contour curves are close together, they will be approximately parallel, so that the grid element will be approximately a small parallelogram, and



(13) $$\Delta A \approx \text{area of parallelogram PQRS} = |PQ \times PR|$$

In the uv-system, the points P, Q, R have the coordinates

$$P : (u_0, v_0), \qquad Q : (u_0 + \Delta u, v_0), \qquad R : (u_0, v_0 + \Delta v) \ ;$$

to use the cross-product however in (13), we need PQ and PR in $\mathbf{i}\,\mathbf{j}$- coordinates. Consider PQ first; we have

(14) $$PQ = \Delta x\,\mathbf{i} + \Delta y\,\mathbf{j} \ ,$$

where Δx and Δy are the changes in x and y as you hold $v = v_0$ and change u_0 to $u_0 + \Delta u$. According to the definition of partial derivative,

$$\Delta x \approx \left(\frac{\partial x}{\partial u}\right)_0 \Delta u, \qquad \Delta y \approx \left(\frac{\partial y}{\partial u}\right)_0 \Delta u;$$

so that by (14),

(15) $$PQ \approx \left(\frac{\partial x}{\partial u}\right)_0 \Delta u\ \mathbf{i} + \left(\frac{\partial y}{\partial u}\right)_0 \Delta u\ \mathbf{j} \ .$$

In the same way, since in moving from P to R we hold u fixed and increase v_0 by Δv,

(16) $$PR \approx \left(\frac{\partial x}{\partial v}\right)_0 \Delta v\ \mathbf{i} + \left(\frac{\partial y}{\partial v}\right)_0 \Delta v\ \mathbf{j} \ .$$

We now use (13); since the vectors are in the xy-plane, $PQ \times PR$ has only a $\mathbf{k}$-component, and we calculate from (15) and (16) that

$$\mathbf{k}\text{-component of } PQ \times PR \approx \begin{vmatrix} x_u \Delta u & y_u \Delta u \\ x_v \Delta v & y_v \Delta v \end{vmatrix}_0$$

(17) $$= \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}_0 \Delta u \Delta v \ ,$$

where we have first taken the transpose of the determinant (which doesn't change its value), and then factored the Δu and Δv out of the two columns. Finally, taking the absolute value, we get from (13) and (17), and the definition (5) of Jacobian,

$$\Delta A \approx \left|\frac{\partial(x, y)}{\partial(u, v)}\right|_0 \Delta u \Delta v \ ;$$

passing to the limit as $\Delta u, \Delta v \to 0$ and dropping the subscript 0 (so that P becomes any point in the plane), we get the desired formula for the area element,

$$dA = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv \ .$$