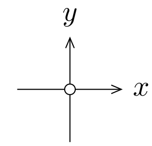
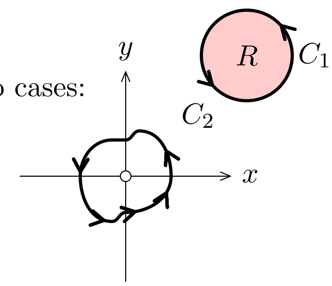
Extended Green's Theorem



Let $\mathbf{F}$ be the "tangential field" $\mathbf{F} = \dfrac{-y\mathbf{i} + x\mathbf{j}}{r^2}$, defined on the punctured plane $D = \mathbf{R}^2 - (0,0)$. It's easy to compute (we've done it before) that $\text{curl}\mathbf{F} = 0$ in D.

Question: For the tangential field $\mathbf{F}$, what do you think the possible values of $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r}$ could be if C were allowed to be any closed curve?

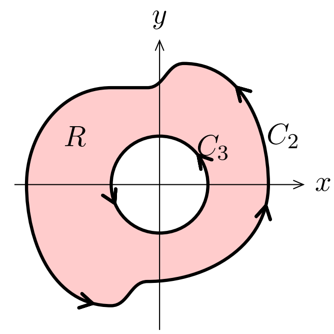


<u>**Answer:**</u> As we saw in lecture, if C is simple and positively oriented we have two cases:
(i) C_1 not around 0 (ii) C_2 around 0

(i) Green's Theorem $\Rightarrow \displaystyle\oint_{C_1} \mathbf{F}\cdot d\mathbf{r} = \iint_R \text{curl}\mathbf{F}\cdot\mathbf{k}\,dA = 0.$

(ii) We show that $\displaystyle\oint_{C_2} \mathbf{F}\cdot d\mathbf{r} = 2\pi.$

Let C_3 be a small circle of radius a, entirely inside C_2.
By extended Green's Theorem

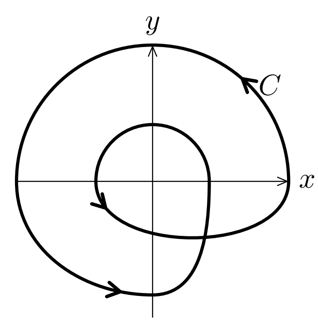


$$\oint_{C_2} \mathbf{F}\cdot d\mathbf{r} - \oint_{C_3} \mathbf{F}\cdot d\mathbf{r} = \iint_R \text{curl}\mathbf{F}\cdot\mathbf{k}\,dA = 0$$

$$\Rightarrow \oint_{C_2} \mathbf{F}\cdot d\mathbf{r} = \oint_{C_3} \mathbf{F}\cdot d\mathbf{r}.$$

On the circle C_3 we can easily compute the line integral:

$$\mathbf{F}\cdot\mathbf{T} = 1/a \;\Rightarrow\; \oint_{C_3} \mathbf{F}\cdot\mathbf{T}\,ds = \int_{C_3} \frac{1}{a}\,ds = \frac{2\pi a}{a} = 2\pi. \quad \text{QED}$$



If C is positively oriented but not simple, the figure to the right suggests that we can break C into two curves around the origin at a point where it crosses itself. Repeating this as often as necessary, we find that $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r} = 2\pi n$, where n is the number of times C goes counterclockwise around (0,0).
If C is negatively oriented $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r} = -\oint_{C'} \mathbf{F}\cdot d\mathbf{r}$, where C' is an oppositely oriented copy of C. Hence, our final answer is that $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r}$ may equal $2\pi n$ for any integer n.

An interesting aside: n is called the *winding number* of C around 0. n also equals the number of times C crosses the positive x-axis, counting $+1$ from below and -1 from above.