Problems: Harmonic Functions and Averages

A function u is called *harmonic* if $\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0$. In this problem we will see that the average value of a harmonic function over any sphere is exactly its value at the center of the sphere.

For simplicity, we'll take the center to be the origin and show the average is $u(0,0,0)$.

Let u be a harmonic function and S_R the sphere of radius R centered at the origin. The average value of u over S is given by $A = \frac{1}{4\pi R^2} \iint_S u(x,y,z)\, dS$.

1. Write this integral explicitly using spherical coordinates.

2. Differentiate A with respect to R

3. Rewrite the formula in part (2) in terms of $\nabla u \cdot \mathbf{n}$.

4. Use the divergence theorem to show $\frac{dA}{dR} = 0$ and conclude the average $A = u(0,0,0)$.