Green's Theorem

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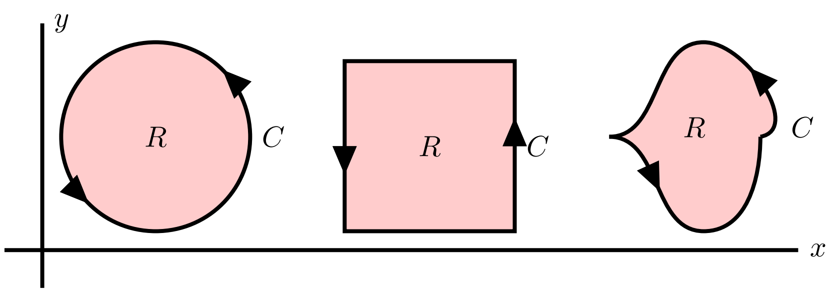
We start with the ingredients for Green's theorem.

(i) C a *simple* closed curve (simple means it never intersects itself)

(ii) R the interior of C.

We also require that C must be *positively oriented*, that is, it must be traversed so its interior is on the left as you move in around the curve. Finally we require that C be *piecewise smooth*. This means it is a smooth curve with, possibly a finite number of corners.

Here are some examples.



Green's Theorem

With the above ingredients for a vector field $\mathbf{F} = \langle M, N\rangle$ we have

$$\oint_C M\,dx + N\,dy = \iint_R N_x - M_y\,dA.$$

We call $N_x - M_y$ the two dimensional curl and denote it $\operatorname{curl}\mathbf{F}$.

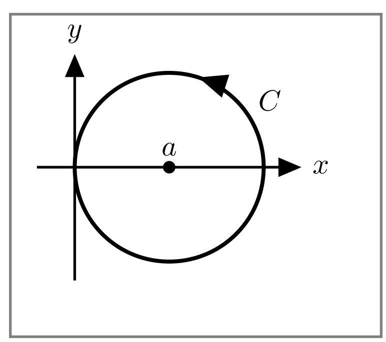
We can write also Green's theorem as

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R \operatorname{curl}\mathbf{F}\,dA.$$

Example 1: (use the right hand side (RHS) to find the left hand side (LHS))

Use Green's Theorem to compute

$I = \oint_C 3x^2y^2\,dx + 2x^2(1+xy)\,dy$ where C is the circle shown.



By Green's Theorem $I = \iint_R 6x^2y + 4x - 6x^2y\,dA = 4\iint_R x\,dA.$

We could compute this directly, but we know $x_{cm} = \dfrac{1}{A}\iint_R x\,dA = a$

$\Rightarrow \iint_R x\,dA = \pi a^3 \Rightarrow \boxed{I = 4\pi a^3.}$

Example 2: (Use the LHS to find the RHS.)

Use Green's Theorem to find the area under one arch of the cycloid

$$x = a(\theta - \sin\theta),\ y = a(1-\cos\theta).$$

The picture shows the curve $C = C_1 - C_2$ surrounding the area we want to find.(Note the minus sign on C_2.)
By Green's Theorem,

$$\oint_C -y\,dx = \iint_R dA = \text{area}.$$

Thus,

$$\text{area} = \oint_{C_1 - C_2} -y\,dx = \int_{C_1} 0 \cdot dx - \int_{C_2} -y\,dx = \int_0^{2\pi} a^2(1-\cos\theta)^2\,d\theta = 3\pi a^2.$$