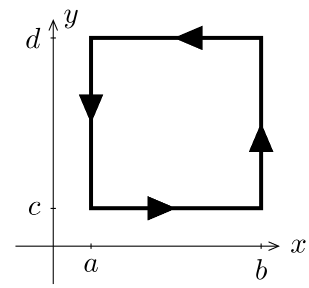
Green's Theorem: Sketch of Proof

Green's Theorem: $\displaystyle\oint_C M\,dx + N\,dy = \iint_R N_x - M_y\,dA.$

Proof:



i) First we'll work on a rectangle. Later we'll use a lot of rectangles to approximate an arbitrary region.

ii) We'll only do $\displaystyle\oint_C M\,dx$ $\left(\displaystyle\oint_C N\,dy\right.$ is similar).

By direct calculation the right hand side of Green's Theorem

$$\iint_R -\frac{\partial M}{\partial y}\,dA = \int_a^b \int_c^d -\frac{\partial M}{\partial y}\,dy\,dx.$$

Inner integral: $\quad -M(x,y)|_c^d = -M(x,d) + M(x,c)$

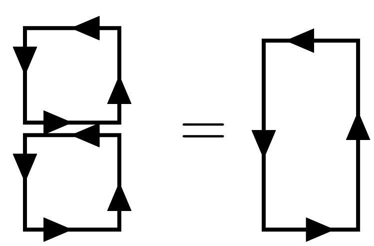
Outer integral: $\quad \displaystyle\iint_R -\frac{\partial M}{\partial y}\,dA = \int_a^b M(x,c) - M(x,d)\,dx.$

For the LHS we have

$$\begin{aligned}\oint_C M\,dx &= \int_{bottom} M\,dx + \int_{top} M\,dx \quad \text{(since } dx = 0 \text{ along the sides)}\\ &= \int_a^b M(x,c)\,dx + \int_b^a M(x,d)\,dx = \int_a^b M(x,c) - M(x,d)\,dx.\end{aligned}$$

So, for a rectangle, we have proved Green's Theorem by showing the two sides are the same.

In lecture, Professor Auroux divided R into "vertically simple regions". This proof instead approximates R by a collection of rectangles which are especially simple both vertically and horizontally.



For line integrals, when adding two rectangles with a common edge the common edges are traversed in opposite directions so the sum is just the line integral over the outside boundary. Similarly when adding a lot of rectangles: everything cancels except the outside boundary. This extends Green's Theorem on a rectangle to Green's Theorem on a sum of rectangles. Since any region can be approximated as closely as we want by a sum of rectangles, Green's Theorem must hold on arbitrary regions.