18.03 Recitation 21, April 27, 2010

First order linear systems

Vocabulary/Concepts: system of differential equations; linear, time-independent, homogeneous; matrix, matrix multiplication; solution, trajectory, phase portrait; companion matrix.

11. Practice in matrix multiplication: Compute the following products:

$$\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} x & y \end{bmatrix}, \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x & u \\ y & v \end{bmatrix}.$$

$$[x+2y], \begin{bmatrix} x & y \\ 2x & 2y \end{bmatrix}, \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}, \begin{bmatrix} x+2y & u+2v \\ 3x+4y & 3u+4v \end{bmatrix}.$$

2. Multiplying by a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ sends a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ to another vector $A\begin{bmatrix} x \\ y \end{bmatrix}$. This operation can be visualized by thinking about where it sends the square with corners $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{i}+\mathbf{j} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For each of the following matrices A, draw segments connecting the dots $\mathbf{0}$, $A\mathbf{i}$, $A(\mathbf{i}+\mathbf{j})$, $A\mathbf{j}$, $\mathbf{0}$, and invent verbal description or name for the operation.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$: holding x-direction unchanged, but lengthening y-direction by a factor of 2.

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$: holding the bottom two vertices on x-axis fixed, but moving the upper two vertices hortizontally to the right by unit 1.

$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$:keeping the dimensions unchanged, but being reflected with respect to x-axis..

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$: keeping the dimensions unchanged, but being reflected with respect to the line $y = x$.

$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$: holding the vertex at the origin fixed, first rotating the square 45 degrees clockwise, then flipping it with respect to x-axis, and finally streching the four sides to the length of $\sqrt{2}$.

3. What is the companion matrix A of the second order equation $\ddot{x} + 2\dot{x} + 2x = 0$? Find two independent real solutions of this second order equation. Let $x_1(t)$ denote the solution with initial condition $x_1(0) = 0$, $\dot{x}_1(0) = 1$. Find it, and then write down the corresponding solution $\mathbf{u_1}(t) = \begin{bmatrix} x_1(t) \\ \dot{x}_1(t) \end{bmatrix}$ of the equation $\dot{\mathbf{u}} = A\mathbf{u}$. What is $\mathbf{u_1}(0)$? Sketch the graphs of $x_1(t)$ and of $\dot{x}_1(t)$, and sketch the trajectory of the solution $\mathbf{u_1}(t)$. Compare these pictures.

Sketch a few more trajectories to fill out the phase portrait. In particular sketch the trajectory of $\mathbf{u_2}(t)$ with $\mathbf{u_2}(0) = \mathbf{i}$.

When trajectories of this companion equation cross the x axis, at what angle do they cross it?

The companion matrix is $\begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$. The characteristic polynomial is $p(s) = s^2 + 2s + 2$ with roots $s = -1 \pm i$, so two independent complex solutions are $e^{(-1+i)t}$ and $e^{(-1-i)t}$. We can combine them to form independent real solutions $e^{-t}\cos t$ and $e^{-t}\sin t$. Considering the given initial condition, we choose $x_1(t) = e^{-t}\sin t$, so $\mathbf{u_1}(t) = \begin{bmatrix} e^{-t}\sin t \\ e^{-t}(\cos t - \sin t) \end{bmatrix}$.

x_1 has envelope $\pm e^{-t}$, which decays exponentially. The graph of x_1 oscillates inside the envelope, and it touches the envelope at odd multiples of $\pi/2$. $\dot{x}_1$ has envelope $\pm\sqrt{2}e^{t}$, and the graph touches the envelope when t has the form $\frac{4k-1}{4}\pi$. The trajectory is an inward spiral, elongated in the northwest-southeast direction. When trajectories cross the x axis, they cross at an angle of $\pi/2$.

4. Let $a + bi$ be a general complex number. There is a matrix A such that if $(a + bi)(x + yi) = (v + wi)$ then

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} v \\ w \end{bmatrix}$$

Find it. What is it for $a + bi = 2$? For $a + bi = i$? For $a + bi = 1 + i$? Draw the parallelograms discussed in **(2)** for these matrices.

We have $v = ax - by$, $w = ay + bx$, so $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. For $a + bi = 2$, $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, and the parallelogram is a square of length 2. For $a + bi = i$, $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and the parallelogram is a square of length 1, rotated by 90 degrees counterclockwise around the origin. For $a + bi = 1 + i$, $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, and the parallelogram is a square of length $\sqrt{2}$, rotated by 45 degrees counterclockwise around the origin.

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