18.03 Hour Exam II Solutions: March 17, 2010

1. **(a)** The characteristic polynomial is $p(s) = s^2 + s + k = \left(s + \frac{1}{2}\right)^2 + \left(k - \frac{1}{4}\right)$. This has a repeated root when $k = \frac{1}{4}$.

(b) If k is larger, the contents of the square root become negative and the roots become non-real: so underdamped. (Note that this does not require the solution to **(a)**.)

(c) Vanishing twice implies underdamped. The pseudoperiod is 2 (since a damped sinusoid vanishes twice for each period), so $\omega_d = \frac{2\pi}{2} = \pi$. From $p(s) = s^2 + s + k = \left(s + \frac{1}{2}\right)^2 + \left(k - \frac{1}{4}\right)$ we find $\omega_d = \sqrt{k - \frac{1}{4}}$, so $k = \pi^2 + \frac{1}{4}$.

2. **(a)** Variation of parameters: $x = ue^{2t}$. $\dot{x} = (\dot{u} + 2u)e^{2t}$, $\ddot{x} = (\ddot{u} + 4\dot{u} + 4u)e^{2t}$, so $\ddot{x} + x = (\ddot{u} + 4\dot{u} + 5u)e^{2t}$, and u must satisfy $\ddot{u} + 4\dot{u} + 5u = 5t$. Undetermined coefficients: $u_p = at + b$, $\dot{u}_p = a$, $\ddot{u}_p = 0$, so $4a + 5(at + b) = 5t$, $a = 1$, $b = -\frac{4}{5}$: $u_p = t - \frac{4}{5}$, $x_p = (t - \frac{4}{5})e^{2t}$.

(b) The homogeneous equation has general solution $a\cos t + b\sin t$, so the general solution of $\ddot{x} + x = 5te^{2t}$ is $x = y + a\cos t + b\sin t$. $3 = x(0) = y(0) + a = 1 + a$ so $a = 2$. $5 = \dot{x}(0) = \dot{y}(0) + b = 2 + b$ so $b = 3$: $x = y + 2\cos(t) + 3\sin(t)$.

3. **(a)** The complex replacement $\ddot{z} + b\dot{z} + kz = e^{i\omega t}$ has exponential solution $z_p = \dfrac{e^{i\omega t}}{p(i\omega)}$. The amplitude of $\mathrm{Re}(z_p)$ is $\dfrac{1}{|p(i\omega)|}$, so we find what value of k minimizes $|p(i\omega)|$. $p(i\omega) = (k - \omega^2) + bi\omega$, so $k = \omega^2$ minimizes the absolute value. [This is interesting; the spring constant resulting in largest gain is the one resulting in a system whose natural frequency matches the driving frequency, independent of the damping constant.]

(b) $p(s) = s^3 - s = s(s-1)(s+1)$, so the modes are $e^{0t} = 1$, e^t, and e^{-t}. The general solution is $ae^{-t} + b + ce^t$.

4. **(a)** By time invariance and linearity we can suppose the input signal is $\cos(\omega t)$. The complex input is $y_{\mathrm{cx}} = e^{i\omega t}$, and $\ddot{z} + \dot{z} + 6z = 6e^{i\omega t}$ has exponential solution $z_p = \dfrac{6}{p(i\omega)}e^{i\omega t} = \dfrac{6}{p(i\omega)}y_{\mathrm{cx}}$, so the complex gain is $H(\omega) = \dfrac{6}{p(i\omega)} = \dfrac{6}{(6 - \omega^2) + i\omega}$.

(b) $H(2) = \frac{6}{(6-4)+2i} = \frac{3}{1+i}$, so $g(2) = |H(2)| = \frac{3}{\sqrt{2}}$.

(c) $\phi = -\mathrm{Arg}(H)(\omega) = \mathrm{Arg}(1 + i) = \frac{\pi}{4}$.

5. **(a)** If we write $q(t) = 4\cos(2t)$, the new input signal is $4\cos(2t - 1) = q(t - \frac{1}{2})$, so by time-invariance, $x = \frac{1}{2}(t - \frac{1}{2})\sin(2(t - \frac{1}{2}))$ solves the new equation.

(b) By linearity, $x = t\sin(2t)$.

(c) The form of the solution indicates resonance: so $\pm 2i$ are roots of the characteristic polynomial, which must thus be $p(s) = m(s - 2i)(s + 2i) = m(s^2 + 4)$. Thus $b = 0$ and $k = 4m$. By the Exponential Response Formula with resonance, $m\ddot{z} + kz = 4e^{2it}$ has solution $\dfrac{4t}{p'(2i)}e^{2it} = \dfrac{4t}{4mi}e^{2it} = \dfrac{t}{mi}e^{2it}$, so the original equation has solution $\frac{1}{m}t\sin(2t)$. Thus $m = 2$, $b = 0$, $k = 8$.